DYNAMICS OF VORTICES FOR THE COMPLEX GINZBURG-LANDAU EQUATION

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ABSTRACT. We study a complex Ginzburg-Landau equation in the plane, which has the form of a Gross-Pitaevskii equation with some dissipation added. We focus on the regime corresponding to well-prepared unitary vortices and derive their asymptotic motion law.

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1. INTRODUCTION

In this paper, we study the dynamics of vortices for a complex Ginzburg-Landau equation on the plane, namely

$$\frac{\delta}{|\log \varepsilon|}\partial_t u_\varepsilon + \alpha i \partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad (\mathrm{CGL})_\varepsilon$$

where $u_\varepsilon : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a complex valued map. Here δ, α and ε denote positive real parameters, and we will mainly focus on the asymptotics as ε tends to zero while δ and α are kept fixed. Up to a change of scale, we may further assume that $\alpha = 1$, and we set $k_\varepsilon = \frac{\delta}{|\log \varepsilon|}$. The complex Ginzburg-Landau equation $(\mathrm{CGL})_\varepsilon$ reduces to the Gross-Pitaevskii equation when $\delta = 0$ and to the parabolic Ginzburg-Landau equation when $\alpha = 0$. Both the Gross-Pitaevskii and the Ginzburg-Landau equations have been widely investigated in the regime which we will consider (see e.g. [8, 17, 14, 4] for the Gross-Pitaevskii equation and [11, 18, 6] and references therein for the parabolic Ginzburg-Landau equation). Typical functions u_ε in this regime are given explicitely by

$$u_\varepsilon^*(a_i, d_i) := \prod_{i=1}^{l} u_{\varepsilon, d_i}(z - a_i) = \prod_{i=1}^{l} f_{1,d_i}\left(\frac{|z - a_i|}{\varepsilon}\right)\left(\frac{z - a_i}{|z - a_i|}\right)^{d_i},$$

where the points $a_i \in \mathbb{R}^2$, $d_i = \pm 1$, and the functions $f_{1,d_i} : \mathbb{R}^+ \mapsto [0, 1]$ which satisfy $f_{1,d_i}(0) = 0, f_{1,d_i}(+\infty) = 1$ are in some sense optimal profiles. The points a_i are called the vortices of the fields u_ε and the d_i their degrees. This class of functions u_ε is of course not invariant by any of the flows corresponding to these equations, but not far from it[1], and it is in particular possible to define notions of point vortices

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[1]see the notion of well-preparedness in Section 1.2

for solutions of $(\mathrm{CGL})_\varepsilon$, at least in an asymptotic way as $\varepsilon \to 0$, and to study their dynamics. This dynamics is eventually governed by a system of ordinary differential equations, at least before collisions.

Two relevant quantities in the study of vortex dynamics are the Ginzburg-Landau energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} e_\varepsilon(u)\,dx = \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2}\,dx,$$

through its energy density $e_\varepsilon(u)$, and the Jacobian

$$Ju = \frac{1}{2}\mathrm{curl}(u \times \nabla u)$$

through its primitive $j(u) = u \times \nabla u$. In the regime which we will consider, one has

$$\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \rightharpoonup \pi\sum_{i=1}^{l}\delta_{a_i} \quad \text{and} \quad Ju_\varepsilon\,dx \rightharpoonup \pi\sum_{i=1}^{l} d_i\delta_{a_i}$$

as $\varepsilon \to 0$, which describes asymptotically the positions and the degrees of the vortices. The quantity $e_\varepsilon(u_\varepsilon)$ was especially used in the study of the parabolic Ginzburg-Landau equation while $j(u_\varepsilon)$ was used in the study of the Gross-Pitaevskii equation. Here, we will rely on both of them.

In the case of the domain being the entire plane $\mathbb{R}^2$, which we consider here, the reference fields $u_\varepsilon(a_i, d_i)$ have infinite Ginzburg-Landau energy E_ε whenever $d = \sum d_i \neq 0$. In [7], a notion of renormalized energy[2] for such data was introduced in order to solve the Cauchy problem for the Gross-Pitaevskii equation. This notion was later used in [4] in order to study the dynamics of vortices for the Gross-Pitaevskii equation in the plane. Our definition of well-prepared data below and part of the subsequent analysis is borrowed from [4].

The complex Ginzburg-Landau equation $(\mathrm{CGL})_\varepsilon$, either in the plane or in the real line, has been vastly considered in the literature, especially as a model for amplitude oscillation in weakly nonlinear systems undergoing a Hopf bifurcation (see e.g. [2] for a survey paper). The mathematical analysis of vortices for $(\mathrm{CGL})_\varepsilon$ was first sketched in [17], where it was presented as an alternative approach (a regularized version) for the study of the Gross-Pitaevskii equation. We believe however that the conclusion regarding the dynamics of vortices for $(\mathrm{CGL})_\varepsilon$ in [17] is erroneous, and that Theorem 2 yields the corrected version.

After the completion of this work we were informed that Spirn, Kurzke, Melcher and Moser [15] independently obtained similar results concerning the dynamics of vortices for $(\mathrm{CGL})_\varepsilon$ in bounded, simply connected domains.

[2]not to be merged with the notion in [3].

1.1. **Renormalized energy and Cauchy Problem.** As mentioned in the introduction, for $d = \sum d_i \neq 0$ the Ginzburg-Landau energy of $u_\varepsilon^*(a_i, d_i)$ is infinite. It can actually be computed that

$$\int_{\mathbb{R}^2} \frac{|\nabla |u_\varepsilon^*(a_i,d_i)||^2}{2} + \frac{(1-|u_\varepsilon^*(a_i,d_i)|^2)^2}{4\varepsilon^2}\, dz < +\infty,$$

whereas as $|z| \to +\infty$,

$$|\nabla u_\varepsilon^*(a_i,d_i)|^2(z) \sim \frac{d^2}{|z|^2},$$

so that

$$\int_{\mathbb{R}^2} \frac{|\nabla u_\varepsilon^*(a_i,d_i)|^2}{2} = +\infty.$$

The renormalized energy introduced in [7] is obtained by substracting the diverging part of the gradient at infinity. More precisely, given a smooth map U_d such that

$$U_d = \left(\frac{z}{|z|}\right)^d \qquad \text{on } \mathbb{R}^2 \setminus B(0,1),$$

we have as $|z| \to +\infty$

$$|\nabla u_\varepsilon^*(a_i,d_i)|^2 \sim |\nabla U_d|^2$$

and one may define

$$\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i,d_i)) := \lim_{R\to+\infty} \int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) - \frac{|\nabla U_d|^2}{2} < +\infty. \tag{1}$$

This definition extends to a larger class of functions, and is a useful ingredient in solving the Cauchy problem. Following [7], we define

$$\mathcal{V} = \{U \in L^\infty(\mathbb{R}^2,\mathbb{C}),\ \nabla^k U \in L^2,\ \forall k \geq 2,\ (1-|U|^2) \in L^2,\ \nabla|U| \in L^2\}.$$

In particular, the space $\mathcal{V}$ contains all the maps u_ε^* as well as the reference maps U_d. We state below and prove in the Appendix global well-posedness in the class $\mathcal{V} + H^1(\mathbb{R}^2)$[3].

Theorem 1. *Let $u_0 = U + w_0$ be in $\mathcal{V} + H^1(\mathbb{R}^2)$. Then there exists a unique global solution $u(t)$ to* $(\mathrm{CGL})_\varepsilon$ *such that $u(t) \in \{U\} + H^1(\mathbb{R}^2)$. If we write $u(t) = U + w(t)$, then w is the unique solution in $C^0(\mathbb{R}_+, H^1(\mathbb{R}^2))$ to*

$$\begin{cases} (k_\varepsilon + i)\partial_t w = \Delta w + f_U(w) \\ w(0) = w_0, \end{cases} \tag{2}$$

where

$$f_U(w) = \Delta U + \frac{1}{\varepsilon^2}(U+w)(1-|U+w|^2).$$

[3] In [10], the Cauchy problem in local spaces is investigated for a more general class of complex Ginzburg-Landau equations.

In addition, w satisfies

$w \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H^2(\mathbb{R}^2)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^*_+, L^\infty(\mathbb{R}^2)),\ \partial_t w \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^2))$

and

$$w \in C^\infty(\mathbb{R}^*_+, C^\infty(\mathbb{R}^2)).$$

Finally, the functional $E_{\varepsilon,U}(u) := E_{\varepsilon,U}(w)$ defined by

$$E_{\varepsilon,U}(u) = \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2} - \int_{\mathbb{R}^2} \Delta U \cdot w + \int_{\mathbb{R}^2} \frac{(1-|U+w|^2)^2}{4\varepsilon^2}$$

satisfies

$$\frac{d}{dt} E_{\varepsilon,U}(u) = -k_\varepsilon \int_{\mathbb{R}^2} |\partial_t w|^2 \, dx, \qquad \forall t \geq 0.$$

As a matter of fact, it follows from an integration by part that if $u \in \{U\} + H^1(\mathbb{R}^2)$ is as in Theorem 1 and if U satisfies in addition $|\nabla U(x)| \leq \frac{C}{\sqrt{|x|}}$, then

$$E_{\varepsilon,U}(u(t)) \equiv \mathcal{E}_{\varepsilon,U}(u(t)) = \lim_{R\to+\infty} \int_{B(R)} (e_\varepsilon(u(t)) - \frac{|\nabla U|^2}{2}) \, dx.$$

The functions $u^*_\varepsilon(a_i, d_i)$ are not H^1 perturbations one of the other, even for fixed $d = \sum d_i$, unless some algebraic relations involving the a_i's and d_i's hold. In order to handle a class of functions containing them all, it is useful to introduce the following equivalence relation on the set $\mathcal{V}$:

$$\begin{aligned}&\forall U, U' \in \mathcal{V}, \qquad U \sim U' \text{ iff}\\ &\deg_\infty(U) = \deg_\infty(U') \text{ and } |\nabla U|^2 - |\nabla U'|^2 \in L^1(\mathbb{R}^2).\end{aligned}$$

Denoting by $[U]$ the corresponding equivalence class of U, we observe that for any configuration (a_i, d_i) such that $\sum d_i = d$, we have $u^*_\varepsilon(a_i, d_i) \in [U_d]$. Therefore the space $[U_d] + H^1(\mathbb{R}^2)$ contains in particular all H^1 perturbations of all reference maps u^*_ε of degree d at infinity.

For a map u in $[U_d] + H^1(\mathbb{R}^2)$, we may now define

$$\mathcal{E}_{\varepsilon,[U_d]}(u) := \lim_{R\to+\infty} \int_{B(R)} e_\varepsilon(u) - \frac{|\nabla U_d|^2}{2},$$

which is a finite quantity. Moreover, for any solution $u = u(t) \in C^0([U_d] + H^1(\mathbb{R}^2))$, we infer from Theorem 1 that

$$\frac{d}{dt} \mathcal{E}_{\varepsilon,[U_d]}(u) = \frac{d}{dt} \mathcal{E}_{\varepsilon,U}(u) = -k_\varepsilon \int_{\mathbb{R}^2} |\partial_t u|^2.$$

The dissipation of $\mathcal{E}_{\varepsilon,[U_d]}(u(t))$ is therefore exactly the same as the dissipation for the usual Ginzburg-Landau energy in the case of bounded, simply connected domains.

1.2. **Statement of the result.** In the sequel, A_n denotes the annulus $B(2^{n+1}) \setminus B(2^n)$ for $n \in \mathbb{N}$, so that $\mathbb{R}^2 = B(2^{n_0}) \cup (\cup_{n \geq n_0} A_n)$.

Definition 1. *Let $a_1, \dots, a_l$ be l distinct points in $\mathbb{R}^2$, $d_i \in \{-1, +1\}$ for $i = 1, \dots, l$ and set $d = \sum d_i$. Let $(u_\varepsilon)_{0<\varepsilon<1}$ be a family of maps in $[U_d] + H^1(\mathbb{R}^2)$. We say that $(u_\varepsilon)_{0<\varepsilon<1}$ is well-prepared with respect to the configuration (a_i, d_i) if there exist $R = 2^{n_0} > \max |a_i|$ and a constant $K_0 > 0$ such that*[4]

$$\lim_{\varepsilon \to 0} \| J u_\varepsilon - \pi \sum_{i=1}^{l} d_i \delta_{a_i} \|_{W_0^{1,\infty}(B(R))^*} = 0, \tag{WP$_1$}$$

$$\sup_{0<\varepsilon<1} E_\varepsilon(u_\varepsilon, A_n) \leq K_0 \qquad \forall n \geq n_0, \tag{WP$_2$}$$

and

$$\lim_{\varepsilon \to 0} \big(\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i, d_i)) \big) = 0. \tag{WP$_3$}$$

We can now state our main theorem as follows

Theorem 2. *Let $(u_\varepsilon^0)_{0<\varepsilon<1}$ in $[U_d] + H^1(\mathbb{R}^2)$ be a family of well-prepared initial data with respect to the configuration (a_i^0, d_i) with $d_i = \pm 1$, and let $(u_\varepsilon(t))_{0<\varepsilon<1}$ in $C(\mathbb{R}^+, [U_d] + H^1(\mathbb{R}^2))$ be the corresponding solution of $(\mathrm{CGL})_\varepsilon$. Let $\{a_i(t)\}_{\{i=1,\dots,l\}}$ denote the solution of the ordinary differential equation*

$$\begin{cases} \pi \dot{a}_i(t) = C_i \big(\delta d_i \mathbb{I}_2 - \mathbb{J}_2 \big) \nabla_{a_i} W, & C_i = \dfrac{-d_i}{1 + \delta^2} \\ a_i(0) = a_i, & i = 1, \dots, l \end{cases} \tag{3}$$

where

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{J}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and W is the Kirchhoff-Onsager functional defined by

$$W(a_i, d_i) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j|.$$

We denote by $[0, T^*)$ its maximal interval of existence. Then, for every $t \in [0, T^*)$, the family $(u_\varepsilon(t))_{0<\varepsilon<1}$ is well-prepared with respect to the configuration $(a_i(t), d_i)$.

2. Evolution formula for u_ε

In this section, we recall or derive a number of evolution formulae involving quantities related to u_ε which we introduce now.

[4]Here, $E_\varepsilon(u, B) \equiv \int_B e_\varepsilon(u)$.

2.1. **Notations.** Throughout this article, we identify $\mathbb{R}^2$ and $\mathbb{C}$. Given $x = (x_1, x_2) \in \mathbb{R}^2$, we set $x^\perp = (-x_2, x_1)$, which in complex notations reads $x^\perp = ix$. For z and $z' \in \mathbb{C}$, $z \cdot z' = \mathrm{Re}(z\overline{z'})$ denotes the scalar product and $z \times z' = z^\perp \cdot z' = -\mathrm{Im}(z\overline{z'})$ the exterior product of z and z' in $\mathbb{R}^2$. For a map $u : \mathbb{R}^2 \to \mathbb{C}$, we denote by

$$j(u) = u \times \nabla u = iu \cdot \nabla u = u^\perp \cdot \nabla u$$

the linear momentum and

$$J(u) = \partial_1 u \times \partial_2 u = \det(\nabla u)$$

the Jacobian of u. For $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, it can be checked that $J(u) = \frac{1}{2}\mathrm{curl} j(u)$ in the distribution sense. On the set where u does not vanish, we have for $k = 1, 2$

$$\partial_k u = \partial_k u \cdot \frac{u}{|u|}\frac{u}{|u|} + \partial_k u \cdot \frac{iu}{|u|}\frac{iu}{|u|}.$$

This yields

$$\partial_k u = \partial_k |u| \frac{u}{|u|} + \frac{j_k(u)}{|u|}\frac{u^\perp}{|u|}, \tag{2.1}$$

hence we have

$$\partial_k u \cdot \partial_l u = \partial_k |u| \partial_l |u| + \frac{j_k(u) j_l(u)}{|u|^2} \tag{2.2}$$

and it follows that

$$|\nabla u|^2 = |\nabla |u||^2 + \frac{|j(u)|^2}{|u|^2}. \tag{2.3}$$

The Hopf differential of u is defined as

$$\omega(u) = |\partial_1 u|^2 - |\partial_2 u|^2 - 2i \partial_1 u \cdot \partial_2 u = 4\partial_z u \overline{\partial_{\overline{z}} u}.$$

It follows from (2.2) that $\omega(u)$ may be rewritten in terms of the components of $\nabla |u|$ and $j(u)$ as

$$\begin{aligned} \omega(u) = {} & \partial_1 |u|^2 - \partial_2 |u|^2 - 2i\, \partial_1 |u| \partial_2 |u| \\ & + \frac{1}{|u|^2}\big(j_1^2(u) - j_2^2(u) - 2i\, j_1(u) j_2(u)\big). \end{aligned} \tag{2.4}$$

We recall that the Ginzburg-Landau energy density is defined by

$$e_\varepsilon(u) = \frac{|\nabla u|^2}{2} + \frac{(1 - |u|^2)^2}{4\varepsilon^2} = \frac{|\nabla u|^2}{2} + V(u),$$

and we set

$$\mu_\varepsilon(u) = \frac{e_\varepsilon(u)}{|\log \varepsilon|}.$$

In view of (2.3), we then have

$$e_\varepsilon(u) = e_\varepsilon(|u|) + \frac{|j(u)|^2}{|u|^2}. \tag{2.5}$$

Finally, we write the right-hand side in $(\mathrm{CGL})_\varepsilon$ as

$$\nabla E(u) = \nabla E_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2).$$

2.2. **Evolution formulae involving the Jacobian and the energy density.** For a smooth map u in space-time, direct computations by integration by part yield for the energy

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} e_\varepsilon(u)\varphi\,dx = -&\int_{\mathbb{R}^2} \partial_t u \cdot \nabla E(u)\varphi\,dx \\ &- \int_{\mathbb{R}^2} \nabla\varphi\cdot(\partial_t u\cdot\nabla u)\,dx \end{aligned} \tag{2.6}$$

and for the Jacobian

$$\frac{d}{dt}\int_{\mathbb{R}^2} J(u)\chi\,dx = -\int_{\mathbb{R}^2} \nabla^\perp\chi\cdot(\partial_t u^\perp\cdot\nabla u)\,dx, \tag{2.7}$$

where $\chi, \varphi \in \mathcal{D}(\mathbb{R}^2)$.

Also, for any vector field $\vec{X} \in C^1(\mathbb{R}^2, \mathbb{C})$ we have (see e.g. [5])

$$\int_{\mathbb{R}^2} \vec{X}\cdot(\nabla E(u)\cdot\nabla u)\,dx = 2\int_{\mathbb{R}^2} \mathrm{Re}\Big(\omega(u)\frac{\partial\vec{X}}{\partial\bar{z}}\Big)\,dz - \int_{\mathbb{R}^2} V(u)\nabla\cdot\vec{X}\,dx.$$

In particular, the choice of $\vec{X} = \nabla\varphi$ or $\vec{X} = \nabla^\perp\chi = i\nabla\chi$ leads to

$$\int_{\mathbb{R}^2} \nabla\varphi\cdot(\nabla E(u)\cdot\nabla u)\,dx = 2\int_{\mathbb{R}^2} \mathrm{Re}\Big(\omega(u)\frac{\partial^2\varphi}{\partial\bar{z}^2}\Big)\,dz - \int_{\mathbb{R}^2} V(u)\Delta\varphi\,dx$$

and

$$\int_{\mathbb{R}^2} \nabla^\perp\chi\cdot(\nabla E(u)\cdot\nabla u)\,dx = -2\int_{\mathbb{R}^2} \mathrm{Im}\Big(\omega(u)\frac{\partial^2\chi}{\partial\bar{z}^2}\Big)\,dz. \tag{2.8}$$

We next consider a solution u of $(\mathrm{CGL})_\varepsilon$, which is smooth in view of Theorem 1. In this case, $\nabla E(u)$ and $\partial_t u$ are related by

$$\partial_t u = \frac{1}{\alpha_\varepsilon}\nabla E(u) = \beta_\varepsilon\nabla E(u), \tag{2.9}$$

where $\alpha_\varepsilon = \dfrac{\delta}{|\log\varepsilon|} + i = k_\varepsilon + i$. Using (2.9) in (2.6) and (2.7), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} e_\varepsilon(u)\varphi\,dx = -\frac{\delta}{|\log\varepsilon|}\int_{\mathbb{R}^2} |\partial_t u|^2\varphi\,dx - \int_{\mathbb{R}^2} \nabla\varphi\cdot(\beta_\varepsilon\nabla E(u)\cdot\nabla u)\,dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^2} J(u)\chi\,dx = -\int_{\mathbb{R}^2} \nabla^\perp\chi\cdot(i\beta_\varepsilon\nabla E(u)\cdot\nabla u)\,dx.$$

In order to get rid of the terms of the form $\int_{\mathbb{R}^2} \vec{X} \cdot (i\nabla E(u) \cdot \nabla u)$, we compute

$$\frac{d}{dt}\int_{\mathbb{R}^2}(bJ(u)\chi - ae_\varepsilon(u)\varphi)$$

where $\beta_\varepsilon = a + ib$. This yields

$$\frac{d}{dt}\int_{\mathbb{R}^2} bJ(u)\chi - ae_\varepsilon(u)\varphi = (b^2+a^2)\int_{\mathbb{R}^2}\nabla^\perp\chi\cdot(\nabla E\cdot\nabla u) + ak_\varepsilon\int_{\mathbb{R}^2}|\partial_t u|^2\,dx$$
$$+\int_{\mathbb{R}^2}(\nabla\varphi - \nabla^\perp\chi)\cdot(a(a+ib)\nabla E\cdot\nabla u).$$

Since $a = \dfrac{k_\varepsilon}{k_\varepsilon^2+1}$ and $b = \dfrac{-1}{k_\varepsilon^2+1}$, we can multiply (2.10) by k_ε^2+1. Using finally (2.8), we obtain

Proposition 1. *Let u solve* $(\mathrm{CGL})_\varepsilon$ *. Then for all $\varphi, \chi \in \mathcal{D}(\mathbb{R}^2)$,*

$$\frac{d}{dt}\int_{\mathbb{R}^2} J(u)\chi + k_\varepsilon e_\varepsilon(u)\varphi = -k_\varepsilon^2\int_{\mathbb{R}^2}|\partial_t u|^2\varphi + 2\int_{\mathbb{R}^2}\mathrm{Im}\Big(\omega(u)\frac{\partial^2\chi}{\partial\bar{z}^2}\Big)$$
$$+ R_\varepsilon(t,\varphi,\chi,u),$$

where the remainder R_ε is defined by

$$R_\varepsilon(t,\varphi,\chi,u) = -k_\varepsilon\int_{\mathbb{R}^2}(\nabla\varphi - \nabla^\perp\chi)\cdot(\beta_\varepsilon\nabla E(u)\cdot\nabla u)$$

or equivalently

$$R_\varepsilon(t,\varphi,\chi,u) = -k_\varepsilon\int_{\mathbb{R}^2}(\nabla\varphi - \nabla^\perp\chi)\cdot(\partial_t u\cdot\nabla u).$$

Proposition 1 allows to derive formally the motion law for the vortices. Indeed, assume that we have

$$Ju_\varepsilon(t) \to \pi\sum_{i=1}^{l} d_i\delta_{a_i(t)},$$

$$\mu_\varepsilon(u_\varepsilon)(t) \to \pi\sum_{i=1}^{l}\delta_{a_i(t)}$$

and $u_\varepsilon(t)$ is close in some sense to $u_\varepsilon^*(a_i(t),d_i)$, and therefore to $u^*(a_i(t),d_i)$, where

$$u^*(a_i,d_i) = \prod_{i=1}^{l}\left(\frac{z-a_i}{|z-a_i|}\right)^{d_i}.$$

We use Proposition 1 with u formally replaced by $u^*(a_i(t),d_i)$ and with choices of test functions φ and χ which are localized and affine near each point $a_i(t)$ and satisfy $\nabla\varphi = \nabla^\perp\chi$ there, so that both terms

$k_\varepsilon^2 \int_{\mathbb{R}^2} |\partial_t u|^2 \varphi$ and $R_\varepsilon(t, \varphi, \chi, u_\varepsilon)$ vanish in the limit $\varepsilon \to 0$. Using the formula (see [4])

$$2 \int_{\mathbb{R}^2} \mathrm{Im}\left(\omega\big(u^*(a_i(t), d_i)\big) \frac{\partial^2 \chi}{\partial \overline{z}^2} \right) = -\pi \sum_{j \neq i} d_i d_j \frac{(a_i - a_j)^\perp}{|a_i - a_j|^2} \cdot \nabla \chi(a_i),$$

we then obtain that for each i

$$\pi d_i \dot{a}_i(t) \cdot \nabla \chi(a_i) + \delta \pi \dot{a}_i(t) \cdot \nabla \varphi(a_i) = -\pi \sum_{j \neq i} d_i d_j \frac{(a_i - a_j)^\perp}{|a_i - a_j|^2} \cdot \nabla \chi(a_i).$$

Taking into account the fact that $\nabla \varphi(a_i) = \nabla^\perp \chi(a_i)$, we infer that

$$\pi \big(d_i \dot{a}_i(t) - \delta \dot{a}_i{}^\perp(t) \big) \cdot \nabla \chi(a_i) = -\pi \sum_{j \neq i} d_i d_j \frac{(a_i - a_j)^\perp}{|a_i - a_j|^2} \cdot \nabla \chi(a_i),$$

which yields the ODE (3).

In Section 4 and 5, in order to give a rigorous meaning to the previous computations, we will prove the convergence of the Jacobians and of the energy densities to the weighted sum of dirac masses mentioned above, and then show that both the energy dissipation $k_\varepsilon^2 \int_{\mathbb{R}^2} |\partial_t u|^2 \varphi$ and the remainder $R_\varepsilon(t, \varphi, \chi, u_\varepsilon)$ vanish asymptotically when ε tends to zero. Finally, we will establish a control of $\omega(u^*(a_i), d_i) - \omega(u_\varepsilon)$ or equivalently of $\omega(u_\varepsilon^*(a_i), d_i) - \omega(u_\varepsilon)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2 / \{a_i(t)\})$.

3. Some results on the renormalized energy

In this section, we study the link between the energy $\mathcal{E}_{\varepsilon, [U_d]}$ and the usual Ginzburg-Landau energy on large balls. This may be achieved for maps having uniform bounded energy on large annuli by defining a degree at infinity.

3.1. **Energy at infinity and topological degree at infinity.** Let A be the annulus $B(2)/B(1)$. We define

$$T_d = \{u \in H^1(A) \text{ s.t. } \exists B \subset B(u),\ |B| \geq \frac{3}{4},\ \forall r \in B,\ \deg(u, \partial B(r)) = d\}$$

and

$$E_\varepsilon^\Lambda = \{u \in H^1(A) \text{ s.t. } E_\varepsilon(u, A) < \Lambda\}.$$

The topological sector of degree d is then defined as

$$S_{d,\varepsilon}^\Lambda = E_\varepsilon^\Lambda \cap T_d.$$

The following Theorem was proved in [1].

Theorem 3. *For all $\Lambda > 0$, there exists $\varepsilon_\Lambda > 0$ such that for every $0 < \varepsilon < \varepsilon_\Lambda$, we have*

$$E_\varepsilon^\Lambda = \bigcup_{d \in \mathbb{Z}} S_{d,\varepsilon}^\Lambda.$$

In the sequel of this section, we fix $\Lambda > \Lambda_d = \pi d^2 \log(2)$ and we set

$$S_d(\Lambda) \equiv S^{\Lambda}_{d,\varepsilon_\Lambda},$$

so that in particular the map U_d belongs to $S_d(\Lambda)$.

Let $u \in [U_d] + H^1(\mathbb{R}^2)$ and for $k \in \mathbb{N}$, set $u_k : z \in A \mapsto u(2^k z)$. By scaling, we find that for every $0 < \varepsilon < \varepsilon_\Lambda$, the map u_k belongs to $E^{\Lambda}_{\varepsilon_\Lambda}$ for $k \geq k(\varepsilon)$ sufficiently large and therefore to some topological sector $S^{\Lambda}_{d(k),\varepsilon}$. Thanks to the uniform bound for the energy $E_\varepsilon(u_k, A)$ for large k, this degree is necessarily identically equal to d.

Proposition 2 ([4], Corollary 3.1)**.** *Let $d \in \mathbb{Z}$ and $\Lambda > \Lambda_d$. For any $u \in [U_d] + H^1(\mathbb{R}^2)$, there exists an integer $n \in \mathbb{N}^*$ such that for all $k \geq n$, the map $u_k : z \in A \mapsto u(2^k z)$ belongs to the topological sector S_d. We denote by $n(u, \Lambda)$ the smallest integer having this property. The map $u \mapsto n(u) = n(u, \Lambda)$ is continuous.*

We first have the following

Lemma 1. *Let $\Lambda > \Lambda_d$ be given. Let $u \in [U_d] + H^1(\mathbb{R}^2)$ and assume that there exists $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$,*

$$E_{\varepsilon_\Lambda}(u, A_n) < \Lambda.$$

Then we have $n(u, \Lambda) \leq n_0$.

The definition of $n(u)$ allows to obtain a lower bound for $\mathcal{E}_{\varepsilon,[U_d]}$ on annuli.

Lemma 2 ([4], Lemma 3.1)**.** *Let $d \in \mathbb{Z}$ and $u \in [U_d] + H^1(\mathbb{R}^2)$. Then, for any $k \geq n(u)$, we have for $\varepsilon < \varepsilon_\Lambda$*

$$\int_{A_k} [e_\varepsilon(u) - \frac{|\nabla U_d|^2}{2}] \geq -C 2^{-2k} \varepsilon^2.$$

Lemma 3 below provides an upper bound for the Ginzburg-Landau energy on sufficiently large balls in terms of the excess energy $\mathcal{E}_{\varepsilon,[U_d]}(u) - \mathcal{E}_{\varepsilon,[U_d]}(u^*_\varepsilon)$. This will enable us to rely on results holding for the Ginzburg-Landau functional in bounded domains in the proof of Theorem 2.

Lemma 3 ([4], Lemma 3.2)**.** *Let $d \in \mathbb{Z}$, $u \in [U_d] + H^1(\mathbb{R}^2)$, $a_1, \ldots, a_l \in \mathbb{R}^2$ and $d_1, \ldots, d_l \in \mathbb{Z}^*$ such that $d = \sum d_i$. Then, for $k \geq 1 + \max\{\log_1 |a_1|, \ldots, \log_2 |a_l|, n(u)\}$ and $R = 2^k$ we have*

$$\int_{B(R)} e_\varepsilon(u) - e_\varepsilon(u^*_\varepsilon(a_i, d_i)) \leq \mathcal{E}_{\varepsilon,[U_d]}(u) - \mathcal{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i, d_i)) + \frac{C}{R},$$

where C depends only on l and d.

3.2. **Explicit identities for the reference map u^*_ε.** We present here an account of some classical identities for the energy of u^*_ε, which are borrowed from [4].

In the sequel, we consider a configuration (a_i, d_i) with $d_i \in \mathbb{Z}^*$ and we set $d = \sum d_i$. We begin with an explicit expansion near each vortex a_j.

Lemma 4. *For $j \in \{1, \dots, l\}$ and $0 < \varepsilon < 1$,*

$$\int_{B(a_j,r)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \pi d_j^2 \log(\frac{r}{\varepsilon}) + \gamma(|d_j|) + O(\frac{r}{r_a})^2 + O(\frac{\varepsilon}{r})^2$$

where $\gamma(|d_j|)$ is some universal constant.

On the other hand, $u^*_\varepsilon(a_i,d_i)$ behaves as $u^*(a_i,d_i)$ away from the vortices, so its energy on $\Omega_{R,r} = B(R) \setminus \cup B(a_j,r)$ is close to the energy of $u^*(a_i,d_i)$ on $\Omega_{R,r}$ which we can compute explicitely (see [3]). Combining the previous expansions, we obtain

Proposition 3. *Let*

$$r_a = \frac{1}{8}\min_{i\neq j}\{|a_i - a_j|\}, \qquad R_a = \max\{|a_i|\}.$$

Then for $R > R_a + 1$, we have as $\varepsilon \to 0$

$$\int_{B(R)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \pi \sum_{i=1}^{l} d_i^2 |\log\varepsilon| + W(a_i,d_i) + \sum_{i=1}^{l}\gamma(|d_i|) + \pi d^2 \log R + O(\frac{R_a}{R}) + o_\varepsilon(1).$$

We observe that as $R \to +\infty$, we have $\pi \log^2 R \sim \int_{B(R)} \frac{|\nabla U_d|^2}{2}$. This yields the following expansion for the renormalized energy

Corollary 1. *When $\varepsilon \to 0$, the following holds*

$$\mathcal{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i,d_i)) = \pi \sum_{i=1}^{l} d_i^2 |\log\varepsilon| + W(a_i,d_i) + \sum_{i=1}^{l}\gamma(|d_i|) - \int_{B(1)} \frac{|\nabla Ud|^2}{2} + o_\varepsilon(1).$$

Concerning the energy on annuli, we finally quote the following result

Lemma 5. *For $R > R_a$, we have*

$$\int_{B(2R)/B(R)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \pi d^2 \log 2 + O\big(\frac{R_a}{R}\big)$$

or, in view of the properties of U_d at infinity,

$$\int_{B(2R)/B(R)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \int_{B(2R)/B(R)} \frac{|\nabla U_d|^2}{2} + O\big(\frac{R_a}{R}\big).$$

4. Coercivity

In this section, we supplement some results from [4] and [13] with estimates which we will later need. These results establish precise estimates in various norms for maps u being close to $u_\varepsilon^*(a_i, d_i)$ in terms of the excess energy with respect to the configuration (a_i, d_i). For a map $u \in [U_d] + H^1(\mathbb{R}^2)$ and a given configuration (a_i, d_i) with $d_i = \pm 1$, we define this excess energy Σ_ε as

$$\Sigma_\varepsilon = \Sigma_\varepsilon(a_i, d_i) = \mathcal{E}_{\varepsilon,[U_d]}(u) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i, d_i)).$$

We also set

$$r_a = \frac{1}{8}\min_{i \neq j}\{|a_i - a_j|\}, \qquad R_a = \max_{i=1,\ldots,l}\{|a_i|\}.$$

Theorem 4. *Let $r \le r_a$ and $2^{n_0} = R_0 > R_a$ such that $\cup_{i=1}^l B(a_i, r) \subset B(R_0)$. Then there exist ε_0 and η_0 depending only on l, r, r_a, R_a, R_0 satisfying the following property. For all $u \in [U_d] + H^1(\mathbb{R}^2)$ such that*

$$\eta = \|Ju - \pi \sum_{i=1}^{l} d_i \delta_{a_i}\|_{W_0^{1,\infty}(B(R))^*} \le \eta_0 \tag{4.1}$$

and

$$2^{n(u)} \le R_0, \tag{4.2}$$

then for $\varepsilon \le \varepsilon_0$ we have

$$\int_{B(R_0)\setminus \cup B(a_i,r)} e_\varepsilon(|u|) + \frac{1}{8}\Big|\frac{j(u)}{|u|} - j(u^*(a_i, d_i))\Big|^2 \le \Sigma_\varepsilon + C(\eta, \varepsilon, \frac{1}{R_0}), \tag{4.3}$$

where C is a continuous function on $\mathbb{R}^3$ vanishing at the origin. Furthermore, there exist points $b_i \in B(a_i, r/2)$ such that

$$\|Ju - \pi \sum_{i=1}^{l} d_i \delta_{b_i}\|_{W_0^{1,\infty}(B(R_0))^*} \le f(R_0, \Sigma_\varepsilon)\varepsilon|\log\varepsilon| \tag{4.4}$$

and

$$\|\mu_\varepsilon(u) - \pi \sum_{i=1}^{l} \delta_{b_i}\|_{W_0^{1,\infty}(B(R_0))^*} \le \frac{g(R_0, r, r_a, \Sigma_\varepsilon)}{|\log\varepsilon|}, \tag{4.5}$$

where f and g are continuous functions on $\mathbb{R}^2$ and $\mathbb{R}^4$.

Proof. Except for the energy concentration (4.5), each of the other statements are already proved in Theorem 6.1 of [4]. We first infer from (4.1) that for all i $\|Ju - \pi d_i \delta_{a_i}\|_{W_0^{1,\infty}(B(a_i,r))^*} \le \eta_0$. If η_0 is sufficiently small with respect to r this gives in view of Theorem 3 in [13] $K_0^i \ge C(r)$, where K_0^i is the local excess energy near the vortex i defined by $K_0^i = \int_{B(a_i,r)} e_\varepsilon(u) - \pi\log(\frac{r}{\varepsilon})$. It follows that

$$\int_{B(a_i,r)} e_\varepsilon(u) \le \int_{B(R_0)} e_\varepsilon(u) - \pi(l-1)|\log\varepsilon| - C(r).$$

On the other hand, since $n(u) \leq n_0$, we have according to Lemma 3 and Proposition 3

$$\int_{B(R_0)} e_\varepsilon(u) \leq \int_{B(R_0)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) + \Sigma_\varepsilon + \frac{C}{R_0} \leq \pi l |\log\varepsilon| + \Sigma_\varepsilon + C.$$

This first implies that $K_0^i \leq C + \Sigma_\varepsilon$. Also, replacing r by $3r/4$ we see that $\int_{B(R_0)\setminus \cup B(a_i,3r/4)} \mu_\varepsilon(u) \leq (C+\Sigma_\varepsilon)|\log\varepsilon|^{-1}$, where C only depends on R_0, r, r_a, R_a.

Now, according to Theorem 2' in [13], the energy density $\mu_\varepsilon(u)$ on $B(a_i,r)$ concentrates at the point $b_i \in B(a_i, r/2)$ where $J(u_\varepsilon)$ concentrates. From Theorem 3.2.1 in [9] and the estimate for K_0^i it follows that

$$\|\mu_\varepsilon(u) - \pi\delta_{b_i}\|_{W_0^{1,\infty}(B(a_i,r))^*} \leq \frac{f(\Sigma_\varepsilon, C)}{|\log\varepsilon|}.$$

Combining the above and the upper bound for the energy density outside the vortex balls finally yields (4.5). $\square$

5. Convergence to Lipschitz vortex paths

In this section, we establish compactness for the Jacobians and the energy densities under weaker assumptions on the initial excess energy. Instead of assuming that this excess energy vanishes initially, we only require that it is uniformly bounded with respect to ε.

Theorem 5. *Let (a_i^0, d_i) with $d_i = \pm 1$ be a configuration of vortices. Let $R = 2^{n_0}$ and $(u_\varepsilon^0)_{0<\varepsilon<1}$ in $[U_d] + H^1(\mathbb{R}^2)$ such that*

$$\lim_{\varepsilon\to 0} \|Ju_\varepsilon - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0}\|_{W_0^{1,\infty}(B(R))^*} = 0, \tag{WP$_1$}$$

$$\sup_{0<\varepsilon<1} E_\varepsilon(u_\varepsilon, A_n) \leq K_0, \qquad \forall n \geq n_0, \tag{WP$_2$}$$

and

$$\sup_{0<\varepsilon<1} \Big(\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i,d_i))\Big) \leq K_1. \tag{WP$_{3'}$}$$

Then there exist $R' = 2^{n_1}$ and $T > 0$ depending only on K_1, R, r_a and R_a, a sequence $\varepsilon_k \to 0$ and l Lipschitz paths $b_i : [0,T] \to \mathbb{R}^2$ starting from a_i^0 such that

$$\sup_{t\in[0,T]} \|Ju_{\varepsilon_k}(t) - \pi\sum_{i=1}^{l} d_i \delta_{b_i(t)}\|_{W_0^{1,\infty}(B(R'))^*} \to 0, \qquad k \to +\infty \tag{5.1}$$

and

$$\sup_{t\in[0,T]} \|\mu_{\varepsilon_k}(u_{\varepsilon_k})(t) - \pi\sum_{i=1}^{l} \delta_{b_i(t)}\|_{W^{1,\infty}(B(R'))^*} \to 0, \qquad k \to +\infty. \tag{5.2}$$

Moreover, there exist a constant $C_0 > 0$ depending only on r_a, R, K_1 and K_0 and a constant $C_1 > 0$ depending on r_a, R and K_1 such that for all $t \in [0, T]$ and for $k \in \mathbb{N}$,

$$E_{\varepsilon_k}(u_{\varepsilon_k}(t), A_n) \leq C_0, \qquad \forall n \geq n_1 \tag{5.3}$$

and

$$\mathcal{E}_{\varepsilon_k,[U_d]}\left(u_{\varepsilon_k}(t)\right) - \mathcal{E}_{\varepsilon_k,[U_d]}\left(u^*_{\varepsilon_k}(b_i(t), d_i)\right) \leq C_1. \tag{5.4}$$

Proof. The proof is very similar to the proof of Theorem 4 in [4]. In the sequel, C will stand for a constant depending only on r_a, R, R_a and K_1.

We first consider $\Lambda > K_0$. Thanks to Lemma 1 and (WP$_2$), there exists $\varepsilon_\Lambda > 0$ such that for all $\varepsilon < \varepsilon_\Lambda$, we have $n(u^0_\varepsilon) = n(u^0_\varepsilon, \Lambda) \leq n_0$. We fix such a Λ and from now on only consider $\varepsilon < \varepsilon_\Lambda$.

We next introduce $R' = \max(R, R_a + r_a)$ and define $n_1 \geq n_0$ as the smallest integer for which $2^{n_1} \leq R'$. In the remainder of the proof, we will assume without loss of generality that $R' = 2^{n_1}$ and we will write $\|\cdot\|$ instead of $\|\cdot\|_{W^{1,\infty}_0(B(R'))^*}$. Our aim is to apply Theorem 4 to each $u_\varepsilon(t)$ for the choice $r = r_a$ and $R_0 = R'$. Let η_0 and ε_0 be the constants provided by Theorem 4 for this choice. First, thanks to (WP$_2$) and (WP$_{3'}$) it turns out that the convergence in (WP$_1$) still holds on the larger ball $B(R')$ (see the proof of Lemma 7.3 in [4]). Therefore, since $t \mapsto Ju_\varepsilon(t) \in L^1(B(R'))$ is continuous for each ε, there exists a time $T_\varepsilon > 0$ such that

$$\|Ju_\varepsilon(s) - \pi \sum_{i=1}^{l} d_i \delta_{a^0_i}\| < \eta_0, \qquad \forall s \in [0, T_\varepsilon). \tag{5.5}$$

We take T_ε to be the maximum time smaller than T^* having this property, where T^* is defined in Theorem 2.

On the other hand, since $t \mapsto E_\varepsilon(u_\varepsilon(t), A_n)$ is continuous uniformly with respect to n and $\Lambda > K_0$, we infer from (WP$_2$) that there exists $T'_\varepsilon > 0$ such that for $s \in [0, T'_\varepsilon]$

$$E_\varepsilon(u_\varepsilon(s), A_n) < \Lambda, \quad \forall n \geq n_1,$$

so according to Lemma 1 we have $n(u_\varepsilon(s)) \leq n_1$ for $s \in [0, T'_\varepsilon]$.

We claim that there exists a constant D depending on K_1, r_a, R and K_0 such that for all $s \in [0, \min(T_\varepsilon, T'_\varepsilon))$,

$$E_\varepsilon(u_\varepsilon(s), A_n) \leq D, \qquad \forall n \geq n_1. \tag{5.6}$$

Consequently, if we assume from the beginning that

$$\Lambda > \max(K_0, D),$$

then it follows from Lemma 1 that $n(u_\varepsilon(s)) \leq n_1$ on $[0, \min(T_\varepsilon, T'_\varepsilon)]$. Therefore $T'_\varepsilon > T_\varepsilon$ and the topological degrees of the maps $u_\varepsilon(t)$ at

infinity remain uniformly bounded by n_1 as long as their Jacobians satisfy (5.5).

Proof of (5.6). As in [4], we decompose for each $n \geq n_1$ $E_\varepsilon(u_\varepsilon(t), A_n) - E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_n)$ as

$$\sum_{\substack{k=n_1 \\ k\neq n}}^{+\infty} \left(E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_k) - E_\varepsilon(u_\varepsilon(t), A_k)\right)$$
$$+ E_\varepsilon\left(u_\varepsilon^*(a_i^0, d_i), B(R')\right) - E_\varepsilon\left(u_\varepsilon(s), B(R')\right)$$
$$+ \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(s)\right) - \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, d_i)\right).$$

We first handle each term of the sum in the right-hand side. In view of Lemmas 2 and 5, we have for $k \geq n_1$

$$\begin{aligned} E_\varepsilon(u_\varepsilon(t), A_k) &\geq -C\varepsilon^2 2^{-2k} + \int_{A_k} \frac{|\nabla U_d|^2}{2} \\ &\geq E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_k) - C(R_a)2^{-k} - C\varepsilon^2 2^{-2k}, \end{aligned}$$

so we deduce that

$$\sum_{\substack{k=n_1 \\ k\neq n}}^{+\infty} \left(E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_k) - E_\varepsilon(u_\varepsilon(t), A_k)\right) \leq C.$$

Next, we infer from the definition of T_ε and Theorem 3 in [13] that $\int_{B(a_i^0, r_a)} e_\varepsilon(u_\varepsilon(s)) \geq \pi|\log\varepsilon| - C$. Observe that R' is chosen so that $\cup B(a_i^0, r_a) \subset B(R')$, so this leads to

$$E_\varepsilon\left(u_\varepsilon(s), B(R')\right) \geq \pi l|\log\varepsilon| - C.$$

Using Proposition 3, we thus find

$$E_\varepsilon\left(u_\varepsilon^*(a_i^0, d_i), B(R')\right) - E_\varepsilon\left(u_\varepsilon(s), B(R')\right) \leq C. \tag{5.7}$$

Finally, we define $\Sigma_\varepsilon^0(s) := \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(s)\right) - \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, d_i)\right)$. Since $\mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(t)\right)$ is non-increasing, we obtain in view of ($\mathrm{WP}_{3'}$)

$$\Sigma_\varepsilon^0(s) \leq \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^0) - \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, d_i)\right) \leq K_1,$$

and (5.6) follows.

We may now apply Theorem 4 to each $u_\varepsilon(t)$ on $[0, T_\varepsilon]$. This provides points $b_i^\varepsilon(s) \in B(a_i^0, \frac{r_a}{2})$ for $0 \leq s \leq T_\varepsilon$. Since $\Sigma_\varepsilon^0(s) \leq K_1$, estimate (4.3) turns into

$$\int_{\Omega_{R',r_a}} e_\varepsilon(|u_\varepsilon(s)|) + \frac{1}{8}\left|\frac{j(u_\varepsilon(s))}{|u_\varepsilon(s)|} - j(u^*(a_i^0, d_i))\right|^2 \leq C,$$

where $\Omega_{R',r_a} = B(R') \setminus \cup B(a_i^0, r_a)$. Also, we have by (2.4) and (2.5)

$$\int_{\Omega_{R',r_a}} e_\varepsilon(u_\varepsilon(s)) \leq C \tag{5.8}$$

and

$$\|\omega(u_\varepsilon(s))\|_{L^1(\Omega_{R',r_a})} \leq C, \tag{5.9}$$

where $C = C(R, r_a, K_1)$. For notation convenience, we may now write μ_ε instead of $\mu_\varepsilon(u_\varepsilon)$.

In the sequel, given any configuration (a_i, d_i), we denote by $\mathcal{H}(a_i)$ the set of functions $\chi, \varphi \in \mathcal{D}(\mathbb{R}^2)$ such that

$$\chi = \sum_{i=1}^{l} \chi_i, \qquad \varphi = \sum_{i=1}^{l} \varphi_i,$$

where for all i

$$\chi_i, \varphi_i \in \mathcal{D}\Big(B(a_i, \frac{3r_a}{2})\Big), \qquad \nabla\varphi_i = \nabla^\perp \chi_i \ \text{ on } \ B(a_i, r_a)$$

and χ_i (hence φ_i) is affine on $B(a_i, r_a)$ with $|\nabla\chi_i(a_i)| = |\nabla\varphi_i(a_i)| \leq 1$.

By definition of r_a such functions χ and φ always exist, and we can moreover estimate their L^∞ norms by

$$\|D\varphi\|_\infty, \|D\chi\|_\infty \leq \frac{C}{r_a}, \qquad \|D^2\varphi\|_\infty, \|D^2\chi\|_\infty \leq \frac{C}{r_a^2}.$$

We next establish a control of the remainder terms appearing in Proposition 1.

Lemma 6. *Assume that* $\sup_{0<\varepsilon<1} T_\varepsilon = T_*$ *is finite. Then there exists a constant* $C = C(r_a, R, K_1, T_*)$ *such that*

$$\int_0^{T_\varepsilon} \int_{\mathbb{R}^2} \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|^2}\, ds \leq \frac{C}{|\log\varepsilon|}$$

and for all $\chi, \varphi \in \mathcal{H}(a_i^0)$

$$\left| \int_0^{T_\varepsilon} \int_{\mathbb{R}^2} (\nabla^\perp\chi - \nabla\varphi) \cdot \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log\varepsilon|}\, ds \right| \leq \frac{C}{|\log\varepsilon|^{\frac{1}{2}}}.$$

Proof. In order to prove the first inequality, we use Theorem 1 and obtain

$$\begin{aligned} \frac{\delta}{|\log\varepsilon|} \int_0^{T_\varepsilon} \int_{\mathbb{R}^2} |\partial_t u_\varepsilon|^2 =& \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^0\right) - \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(T_\varepsilon)\right) \\ \leq& K_1 + \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, d_i)\right) - \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(T_\varepsilon)\right). \end{aligned}$$

Since $n(u_\varepsilon(T_\varepsilon)) \leq n_1$ we have by Lemma 3

$$\begin{aligned} \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, di)\right) -& \mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon(T_\varepsilon)\right) \\ &\leq \int_{B(R')} e_\varepsilon\left(u_\varepsilon^*(a_i^0, d_i)\right) - \int_{B(R')} e_\varepsilon\left(u_\varepsilon(T_\varepsilon)\right) + \frac{C}{R'} \end{aligned}$$

which is bounded in view of (5.7). It then suffices to divide all terms by $|\log\varepsilon|$.

For the second assertion, we set $\xi = \nabla^\perp \chi - \nabla \varphi$ which has compact support in $A = \cup A_i$, where $A_i = B(a_i^0, \frac{3r_a}{2}) \setminus B(a_i^0, r_a)$, and we apply Cauchy-Schwarz inequality. We obtain

$$\left(\int_0^{T_\varepsilon} \int_{\mathbb{R}^2} (\nabla^\perp \chi - \nabla \varphi) \cdot \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|} \right)^2 \\ \leq \left(\int_0^{T_\varepsilon} \int_{\mathbb{R}^2} \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|^2} \right) \cdot \left(\int_0^{T_\varepsilon} \int_A |\nabla u_\varepsilon|^2 |\xi|^2 \right).$$

Since $A \subset \Omega_{R', r_a}$, we infer from (5.8)

$$\int_0^{T_\varepsilon} \int_A |\nabla u_\varepsilon|^2 |\xi|^2 \leq \|\xi\|_\infty^2 \int_0^{T_\varepsilon} \int_A |\nabla u_\varepsilon|^2 \leq C T_* \|\xi\|_\infty^2,$$

and the conclusion finally follows from the first part of the proof. □

We may now establish the following

Lemma 7. *There exists* $T = T(r_a, R_a, R, K_1) > 0$ *such that*

$$\liminf_{\varepsilon \to 0} T_\varepsilon \geq T.$$

Proof. The first step consists in showing that for $(\chi, \varphi) \in \mathcal{H}(a_i^0)$, for $s, t \in [0, T_\varepsilon]$ and $i = 1, \ldots, l$ we have

$$\left| \langle \chi_i, J u_\varepsilon(t) - J u_\varepsilon(s) \rangle + \delta \langle \varphi_i, \mu_\varepsilon(t) - \mu_\varepsilon(s) \rangle \right| \\ \leq C |t - s| + \frac{C}{|\log \varepsilon|^{\frac{1}{2}}}. \tag{5.10}$$

Indeed, we fix i and we invoke Proposition 1 for $u \equiv u_\varepsilon$ and the choice of test functions (χ_i, φ_i). Integrating (2.10) on $[s, t]$ yields

$$|\langle \chi_i, J u_\varepsilon(t) - J u_\varepsilon(s) \rangle + \delta \langle \varphi_i, \mu_\varepsilon(t) - \mu_\varepsilon(s) \rangle| \leq 2 \int_s^t \int \left| \mathrm{Im} \left(\omega(u_\varepsilon) \frac{\partial^2 \chi_i}{\partial \bar{z}^2} \right) \right| \\ + \int_s^t \int \left| \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|^2} \varphi_i + (\nabla^\perp \chi_i - \nabla \varphi_i) \cdot \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|} \right|,$$

where $\frac{\partial^2 \chi_i}{\partial \bar{z}^2}$ has support in $C_i \subset \Omega_{R', r_a}$, and it finally suffices to use (5.9) and Lemma 6.

In a second step, we take advantage of the equality

$$\| J u_\varepsilon(T_\varepsilon) - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0} \| \equiv \eta_0.$$

We set

$$\nu_{i,\varepsilon} = d_i \frac{b_i^\varepsilon(T_\varepsilon) - a_i^0}{|b_i^\varepsilon(T_\varepsilon) - a_i^0|}, \qquad i = 1, \ldots, l$$

and we define $\chi_{i,\varepsilon}$, $\varphi_{i,\varepsilon}$ so that for $x \in B(a_i^0, r_a)$,

$$\chi_{i,\varepsilon}(x) = \nu_{i,\varepsilon} \cdot x, \qquad \varphi_{i,\varepsilon}(x) = \nu_{i,\varepsilon}^\perp \cdot x,$$

and we require additionally that $\chi = \sum \chi_{i,\varepsilon}$ and $\varphi = \sum \varphi_{i,\varepsilon}$ belong to $\mathcal{H}(a_i^0)$; we can moreover choose $\varphi_{i,\varepsilon}$ and $\chi_{i,\varepsilon}$ so that their norms in $C^2(B(R))$ remain bounded uniformly in ε. As $b_i^\varepsilon(T_\varepsilon) \in B(a_i^0, r_a/2)$, we have

$$|d_i||b_i^\varepsilon(T_\varepsilon) - a_i^0| = d_i\chi(b_i^\varepsilon(T_\varepsilon) - a_i^0) + \delta\varphi(b_i^\varepsilon(T_\varepsilon) - a_i^0),$$

so that

$$\|\pi \sum_{i=1}^{l} d_i(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0})\| = \langle \pi \sum_{i=1}^{l} d_i(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \chi\rangle + \delta\langle \pi \sum_{i=1}^{l}(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \varphi\rangle.$$

On the other hand, we have

$$\|Ju_\varepsilon(T_\varepsilon) - \pi\sum_{i=1}^{l} d_i\delta_{a_i^0}\| \leq \|Ju_\varepsilon(T_\varepsilon) - \pi\sum_{i=1}^{l} d_i\delta_{b_i^\varepsilon(T_\varepsilon)}\| + \|\pi\sum_{i=1}^{l} d_i(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0})\|.$$

The second term in the right-hand side may be rewritten as

$$\langle \pi \sum_{i=1}^{l} d_i(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \chi\rangle + \delta\langle \pi \sum_{i=1}^{l}(\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \varphi\rangle = A + B + C,$$

where

$$\begin{aligned} A &= \langle \pi\sum_{i=1}^{l} d_i\delta_{b_i^\varepsilon(T_\varepsilon)} - Ju_\varepsilon(T_\varepsilon), \chi\rangle + \delta\langle \pi\sum_{i=1}^{l}\delta_{b_i^\varepsilon(T_\varepsilon)} - \mu_\varepsilon(T_\varepsilon), \varphi\rangle \\ &\leq C\left(\|Ju_\varepsilon(T_\varepsilon) - \sum_{i=1}^{l} d_i\delta_{b_i^\varepsilon(T_\varepsilon)}\| + \delta\|\mu_\varepsilon(T_\varepsilon) - \sum_{i=1}^{l}\delta_{b_i^\varepsilon(T_\varepsilon)}\|\right), \end{aligned}$$

B is given by

$$B = \langle Ju_\varepsilon(T_\varepsilon) - Ju_\varepsilon(0), \chi\rangle + \delta\langle \mu_\varepsilon(T_\varepsilon) - \mu_\varepsilon(0), \varphi\rangle$$

and finally

$$\begin{aligned} C &= \langle Ju_\varepsilon^0 - \pi\sum_{i=1}^{l} d_i\delta_{a_i^0}, \chi\rangle + \delta\langle \mu_\varepsilon(u_\varepsilon^0) - \pi\sum_{i=1}^{l}\delta_{a_i^0}, \varphi\rangle \\ &\leq C\left(\|Ju_\varepsilon^0 - \sum_{i=1}^{l} d_i\delta_{a_i^0}\| + \delta\|\mu_\varepsilon(u_\varepsilon^0) - \sum_{i=1}^{l}\delta_{a_i^0}\|\right). \end{aligned}$$

In view of the bound provided by (5.10) for B, estimates (4.4)- (4.5) and the fact that $\Sigma_\varepsilon^0(s) \leq K_1$ for $0 \leq s \leq T_\varepsilon$, this implies

$$\eta_0 = \|Ju_\varepsilon(T_\varepsilon) - \pi\sum_{i=1}^{l} d_i\delta_{a_i^0}\| \leq C(\varepsilon|\log\varepsilon| + |\log\varepsilon|^{-1} + |\log\varepsilon|^{-\frac{1}{2}}) + CT_\varepsilon,$$

and letting $\varepsilon \to 0$ yields the conclusion. Lemma 7 is proved. $\square$

Proof of Theorem 5 completed.
We consider $t, s \in [0,T]$. Arguing as in the proof of Lemma 7 (with T_ε and 0 replaced by t and s), we find that for all χ, φ belonging to $\mathcal{H}(a_i^0)$

$$\begin{aligned}
&\Big|\sum_{i=1}^{l} d_i \Big[\chi(b_i^\varepsilon(t)) - \chi(b_i^\varepsilon(s))\Big] + \delta\Big[\varphi(b_i^\varepsilon(t)) - \varphi(b_i^\varepsilon(s))\Big]\Big| \\
&\qquad \le C \sup_{\tau\in[0,T]} \Big(\|Ju_\varepsilon(\tau) - \sum_{i=1}^{l} d_i \delta_{b_i^\varepsilon(\tau)}\| + \delta\|\mu_\varepsilon(\tau) - \sum_{i=1}^{l} \delta_{b_i^\varepsilon(\tau)}\|\Big) \\
&\qquad + \Big|\langle Ju_\varepsilon(t) - Ju_\varepsilon(s), \chi\rangle + \delta\langle \mu_\varepsilon(t) - \mu_\varepsilon(s), \varphi\rangle\Big|,
\end{aligned}$$

which is bounded by $o_\varepsilon(1) + c|t-s|$ by (4.4)-(4.5) and (5.10). Considering successively $\chi(x) = e_1 \cdot x$ and $\chi(x) = e_2 \cdot x$ on each $B(a_i^0, r_a)$, we obtain

$$|b_i^\varepsilon(t) - b_i^\varepsilon(s)| \le c|t-s| + o_\varepsilon(1). \tag{5.11}$$

Next, using that $b_i^\varepsilon \in B(a_i^0, r_a)$ and a standard diagonal argument, we may construct a sequence $(\varepsilon_k) \to 0$ and paths $b_i(t)$ such that $b_i^{\varepsilon_k}(t)$ converges to $b_i(t)$ for all $t \in \mathbb{Q} \cap [0,T]$. We infer then from (4.4)-(4.5) that the convergence statements (5.1)-(5.2) in Theorem 5 hold for these times. Moreover, in view of (5.11) these paths are Lipschitz on $[0,T] \cap \mathbb{Q}$, so that they can be extended in a unique way to Lipschitz paths (still denoted by $b_i(t)$) on the whole of $[0,T]$. We can finally establish that the convergence (5.1)-(5.2) holds uniformly with respect to $t \in [0,T]$ by using again (5.11) and (4.4)-(4.5).

Finally, we already know from (5.6) that estimate (5.3) holds for the full family $(u_\varepsilon)_{\varepsilon<\varepsilon_\Lambda}$. In order to show (5.4), we recall first the uniform bound $\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon(t)) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0, d_i)) \le K_1$. On the other hand, Corollary 1 gives

$$\mathcal{E}_{\varepsilon,[U_d]}\left(u_\varepsilon^*(a_i^0, d_i)\right) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(b_i(t), d_i)) = W(a_i^0, d_i) - W(b_i(t), d_i) \le C,$$

since the $b_i's$ are continuous and remain separated on $[0,T]$. This yields the bound (5.4) and concludes the proof of Theorem 5. $\square$

As mentioned in the beginning of the proof of Theorem 5, the convergence of the initial data in (WP$_1$) actually holds on every large ball $B(L)$, $L = 2^n \ge R$, so that we find the same conclusions when replacing R by L.

Lemma 8 ([4], Lemma 7.3)**.** *There exists a subsequence, still denoted by ε_k, such that for all $L \ge 2^{n_1}$,*

$$\eta_k := \sup_{[0,T]} \|Ju_{\varepsilon_k}(t) - \pi\sum_{i=1}^{l} d_i\delta_{b_i(t)}\|_{W_0^{1,\infty}(B(L))^*} \to 0, \qquad k \to +\infty.$$

For $t \in [0,T]$ and sufficiently large $k \in \mathbb{N}$, we may therefore apply Theorem 4 to $u_{\varepsilon_k}(t)$ with respect to the configuration $(b_i(t), d_i)$ and with the choice $R_0 = L = 2^n$ for each $n \ge n_1$. We are led to introduce

the excess energy at time t with respect to the configuration $(b_i(t), d_i)$ by

$$\Sigma_{\varepsilon_k}(t) = \mathcal{E}_{\varepsilon_k,U_d}(u_{\varepsilon_k}(t)) - \mathcal{E}_{\varepsilon_k,U_d}\left(u^*_{\varepsilon_k}(b_i(t), d_i)\right),$$

which is uniformly bounded on $[0, T]$ in view of (5.4). Letting first k, then n tend to $+\infty$, we can get rid of the dependance on R in (4.3).

Lemma 9. *For all $r \le r_a/2$ and $K \ge 2^{n_1}$, we have for sufficiently large k and $t, t_1, t_2 \in [0, T]$*

$$\int_{B(K)\setminus\cup B(b_i(t),r)} e_{\varepsilon_k}(|u_{\varepsilon_k}(t)|) + \frac{1}{8}\Big|\frac{j(u_{\varepsilon_k}(t))}{|u_{\varepsilon_k}(t)|} - j(u^*(b_i(t), d_i))\Big|^2 \le \Sigma_{\varepsilon_k}(t) + C(\varepsilon_k, \eta_k, \frac{1}{K}).$$

Therefore, we have as $k \to +\infty$

$$\limsup_{k\to+\infty} \int_{t_1}^{t_2} \int_{B(K)\setminus\cup B(b_i(t),r)} e_{\varepsilon_k}(|u_{\varepsilon_k}(t)|) + \frac{1}{8}\Big|\frac{j(u_{\varepsilon_k})(t)}{|u_{\varepsilon_k}(t)|} - j(u^*(b_i(t), d_i))\Big|^2 \le \limsup_{k\to+\infty} \int_{t_1}^{t_2} \Sigma_{\varepsilon_k}(t).$$

Consequently, it appears that the distance between $u_{\varepsilon_k}(t)$ and $u^*(b_i(t), d_i)$ may be asymptotically entirely controlled by $\limsup \Sigma_{\varepsilon_k}(t)$.

We now define the trajectory set

$$\mathcal{T} = \{(t, b_i(t)),\ t \in [0, T],\ i = 1, \dots, l\}$$

and

$$\mathcal{G} = [0, T] \times \mathbb{R}^2 \setminus \mathcal{T}.$$

Thanks to the uniform bounds in $L^2_{\mathrm{loc}}(\mathcal{G})$ provided by Lemma 9, we establish the following

Proposition 4. *There exists a subsequence, still denoted ε_k, such that*

$$\frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} \rightharpoonup j(u^*(b_i(\cdot), d_i))$$

weakly in $L^2_{\mathrm{loc}}(\mathcal{G})$ as $k \to +\infty$.

Proof. Let B be any bounded subset of $\mathbb{R}^2$. First, we observe that according to Lemma 8

$$\mathrm{curl}\Big(j(u_{\varepsilon_k})\Big) = 2Ju_{\varepsilon_k} \to 2\pi \sum_{i=1}^{l} d_i \delta_{b_i(\cdot)} = \mathrm{curl}\Big(j(u^*(b_i(\cdot), d_i))\Big) \quad (5.12)$$

in $\mathcal{D}'([0, T] \times B)$.

On the other hand, we have

$$\mathrm{div}\Big(j(u_{\varepsilon_k})\Big) \to 0 = \mathrm{div}\Big(ju^*((b_i(\cdot), d_i))\Big) \quad (5.13)$$

in $\mathcal{D}'([0, T] \times B)$.

Indeed, since u_{ε_k} solves $(\text{CGL})_\varepsilon$ we obtain by considering the exterior product

$$k_{\varepsilon_k} u_{\varepsilon_k} \times \partial_t u_{\varepsilon_k} + u_{\varepsilon_k} \cdot \partial_t u_{\varepsilon_k} = u_{\varepsilon_k} \times \Delta u_{\varepsilon_k} = \operatorname{div}\big(j(u_{\varepsilon_k})\big),$$

so we are led to

$$\operatorname{div}(j u_{\varepsilon_k}) = k_{\varepsilon_k} u_{\varepsilon_k} \times \partial_t u_{\varepsilon_k} + \frac{1}{2}\varepsilon_k \frac{d}{dt}\left(\frac{|u_{\varepsilon_k}|^2 - 1}{\varepsilon_k}\right). \tag{5.14}$$

Now, applying Lemma 3 to u_{ε_k}, we find

$$\sup_{[0,T]} E_{\varepsilon_k}(u_{\varepsilon_k}(t), B) \leq \pi l |\log \varepsilon| + \Sigma_{\varepsilon_k}(t) + C \leq \pi l |\log \varepsilon| + C, \tag{5.15}$$

where the second inequality is itself a consequence of (5.4). This implies first that $|u_{\varepsilon_k}| \to 1$ in $L^2([0,T] \times B)$. Moreover, we infer that the second term in the r.h.s of (5.14) converges to zero in the distribution sense on $[0,T] \times B$. For the first one, it suffices to use Cauchy-Schwarz inequality combined with the L^2 bound provided by Lemma 6 and the already mentioned uniform bounds of $|u_{\varepsilon_k}|$ in L^2_{loc}.

We then infer from Lemma 8 and (5.15) that $j(u_{\varepsilon_k})$ is uniformly bounded in $L^p_{\text{loc}}([0,T] \times \mathbb{R}^2)$ for all $p < 2$. This is e.g. a consequence of Theorem 3.2.1 in [9] and the remarks that follow. We deduce from (5.12) and (5.13) that up to a subsequence, we have

$$j(u_{\varepsilon_k}) \rightharpoonup j_1 = j(u^*(b_i(\cdot), d_i)) + H \tag{5.16}$$

weakly in $L^p_{\text{loc}}([0,T] \times \mathbb{R}^2)$, where H is harmonic in x on $[0,T] \times \mathbb{R}^2$.

On the other hand, it follows from the first part of Lemma 9 that there exists j_2 such that, taking subsequences if necessary, $j(u_{\varepsilon_k})/|u_{\varepsilon_k}| \rightharpoonup j_2$ weakly in $L^2_{\text{loc}}(\mathcal{G})$.

Taking into account the strong convergence $|u_{\varepsilon_k}| \to 1$ in $L^2_{\text{loc}}([0,T] \times \mathbb{R}^2)$, we obtain $j_1 = j_2 \in L^2_{\text{loc}}(\mathcal{G})$. The second part of Lemma 9 combined with (5.16) then yields

$$\|H\|_{L^2_{\text{loc}}(\mathcal{G})} \leq \liminf_{k \to +\infty} \|\frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} - j(u^*(b_i, d_i))\|_{L^2_{\text{loc}}(\mathcal{G})} \leq CT,$$

where C depends only on K_1, R and r_a, so finally $\|H\|_{L^2([0,T] \times \mathbb{R}^2)} \leq CT$. Since H is harmonic in x, we find that $H(t, \cdot)$ is bounded on $\mathbb{R}^2$ for almost every t and therefore is identically zero. We end up with $j_1 = j_2 = j(u^*(b_i(\cdot), d_i))$ in $\mathcal{G}$, and the conclusion follows. $\square$

6. Proof of Theorem 2

In this section, we present the proof of Theorem 2. We let $\{b_i(t)\}$ be the l Lipschitz paths on $[0,T]$ provided by Theorem 5 and $\{a_i(t)\}$ be the unique maximal solution defined on $I = [0, T^*)$ to (3) with initial conditions a_i^0. Our aim is to show that $a_i(t) \equiv b_i(t)$ on I. We will first prove that this holds on $[0,T]$. By Rademacher's Theorem, the time

derivatives $\dot{b}_i(t)$ exist and are bounded almost everywhere on $[0,T]$. Without loss of generality, we may assume $T < T^*$, so that

$$|\dot{a}_i(t)| \le C, \ \ |\dot{b}_i(t)| \le C, \qquad \text{a.e. on } [0,T]. \tag{6.1}$$

Moreover, we may assume, decreasing possibly T, that $|a_i(t) - b_i(t)| \le r_a/2$ for all i. Hence, the trajectories $a_i(t)$ remain in $B(a_i^0, r_a)$ on $[0,T]$. We introduce

$$h(t) = \sum_{i=1}^{l} \int_0^t |\dot{a}_i(s) - \dot{b}_i(s)|\, ds, \qquad \sigma(t) = \sum_{i=1}^{l} |a_i(t) - b_i(t)|,$$

then h is Lipschitz on $[0,T]$ and for almost every $t \in [0,T]$ we have $h'(t) = \sum_{i=1}^{l} |\dot{a}_i(t) - \dot{b}_i(t)|$. Note that since σ is absolutely continuous and $\sigma(0) = 0$, we have for all $t \in [0,T]$

$$\sigma(t) = \int_0^t \sigma'(s)\, ds \le h(t)$$

therefore it suffices to show that h is identically zero on $[0,T]$. This will be done by mean of Gronwall's Lemma.

Lemma 10. *For all $t_1, t_2, t \in [0,T]$, we have*

$$\limsup_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \le Ch(t)$$

and

$$\limsup_{k\to+\infty} \int_{t_1}^{t_2} \Sigma_{\varepsilon_k}(s)\, ds \le C \int_{t_1}^{t_2} h(s)\, ds,$$

where C only depends on r_a, K_0, R_a.

Proof. For $t \in [0,T]$, we decompose $\Sigma_{\varepsilon_k}(t)$ as

$$\Sigma_{\varepsilon_k}(t) = \mathcal{E}_{\varepsilon_k,[U_d]}\left(u_{\varepsilon_k}(t)\right) - \mathcal{E}_{\varepsilon_k,[U_d]}(u^0_{\varepsilon_k}) + \Sigma_{\varepsilon_k}(0) \\ + \mathcal{E}_{\varepsilon_k,[U_d]}\left(u^*_{\varepsilon_k}(a_i^0, d_i)\right) - \mathcal{E}_{\varepsilon_k,[U_d]}\left(u^*_{\varepsilon_k}(b_i(t), d_i)\right).$$

Appealing to Corollary 1 and Theorem 1, we obtain

$$\Sigma_{\varepsilon_k}(t) = -\delta \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|} + \Sigma_{\varepsilon_k}(0) + W(a_i^0, d_i) - W(b_i(t), d_i) + o_{\varepsilon_k}(1).$$

Using that W is Lipschitz away from zero, we estimate the last term as follows

$$W(a_i^0, d_i) - W(b_i(t), d_i) = W(a_i^0, d_i) - W(a_i(t), d_i) + W(a_i(t), d_i) - W(b_i(t), d_i) \\ \le -\int_0^t \sum_{i=1}^{l} \dot{a}_i(s) \cdot \nabla_{a_i} W(s)\, ds + C\sigma(t).$$

Since the a_i solve the Cauchy problem (3), an explicit computation gives

$$\dot{a}_i(s) \cdot \nabla_{a_i} W(s) = \frac{\delta}{\pi} C_i d_i |\nabla_{a_i} W|^2 = -\delta\pi |\dot{a}_i(s)|^2,$$

so that

$$\Sigma_{\varepsilon_k}(t) \le \Sigma_{\varepsilon_k}(0) + \delta\pi \int_0^t \sum_{i=1}^l |\dot a_i(s)|^2\, ds - \delta \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|} + C\sigma(t) + o_{\varepsilon_k}(1).$$

We handle next the energy dissipation in the right-hand side. In view of Lemma 6, we have $\int_{[0,T]\times\mathbb{R}^2} |\partial_t u_{\varepsilon_k}|^2 \le C|\log\varepsilon_k|$, while $E_{\varepsilon_k}(u_{\varepsilon_k}, B(R')) \le \pi l|\log\varepsilon_k| + C$. Applying Corollary 7 in [19] to (u_{ε_k}), we obtain

$$\liminf_{k\to+\infty} \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log\varepsilon_k|} \ge \pi \sum_{i=1}^l \int_0^t |\dot b_i(t)|^2\, ds. \tag{6.2}$$

Now, we have thanks to (6.1)

$$\sum_{i=1}^l \int_0^t \left(|\dot a_i(s)|^2 - |\dot b_i(s)|^2\right) \le C \sum_{i=1}^l \int_0^t |\dot a_i(s) - \dot b_i(s)|\, ds = Ch(t),$$

whereas $\Sigma_{\varepsilon_k}(0) \to 0$ by assumption, hence we get

$$\limsup_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \le C\left(\sigma(t) + h(t)\right).$$

Applying Fatou's Lemma in (6.2) finally also provides the corresponding integral version, and lastly, it suffices to use that $\sigma \le h$. □

As suggested in the introduction, the map $u^*(a_i(t), d_i)$ solves the evolution formula given in Proposition 1 in the asymptotics $\varepsilon \to 0$.

Lemma 11. *We have for $t \in [0,T]$ and $\chi, \varphi \in \mathcal{H}(a_i^0)$*

$$\pi \frac{d}{dt} \sum_{i=1}^l d_i \chi\left(a_i(t)\right) + \delta\varphi\left(a_i(t)\right) = 2\int_{\mathbb{R}^2} \mathrm{Im}\left(\omega\left(u^*(a_i(t), d_i)\right) \frac{\partial^2\chi}{\partial \overline{z}^2}\right).$$

Proof. We use the following formula proved in [5], valid for any configuration (a_i, d_i) and any test function χ which is affine near the point vortices .

$$2\int_{\mathbb{R}^2} \mathrm{Im}\left(\omega(u^*(a_i(t), d_i)) \frac{\partial^2\chi}{\partial\overline{z}^2}\right) = -\pi \sum_{i\neq j} d_i d_j \frac{(a_i(t) - a_j(t))^\perp}{|a_i(t) - a_j(t)|^2} \cdot \nabla\chi(a_i(t)).$$

On the other hand, we compute

$$\begin{aligned} \frac{d}{dt}\left(\sum_{i=1}^l d_i\chi(a_i) + \delta\varphi(a_i)\right) &= \sum_{i=1}^l \Big(d_i \nabla\chi(a_i^0)\cdot \dot a_i(t) + \delta\nabla\varphi(a_i^0)\cdot\dot a_i(t)\Big) \\ &= \sum_{i=1}^l d_i\nabla\chi(a_i^0)\cdot\Big(\dot a_i(t) - \delta d_i \dot a_i{}^\perp(t)\Big), \end{aligned}$$

where the second equality follows from the relation $\nabla\varphi(a_i^0) = \nabla^\perp\chi(a_i^0)$. Next, we deduce from (3)

$$\pi\Big(\dot a_i(t) - \delta d_i \dot a_i{}^\perp(t)\Big) = -C_i(1+\delta^2 d_i^2)\nabla^\perp_{a_i} W = d_i \nabla^\perp_{a_i} W,$$

and we obtain

$$\pi \frac{d}{dt}\left(\sum_{i=1}^{l} d_i\chi(a_i) + \delta\varphi(a_i)\right) = \sum_{i=1}^{l} \nabla\chi(a_i)\cdot\nabla_{a_i}^{\perp} W$$
$$= -\pi\sum_{i\neq j} d_i d_j \frac{(a_i - a_j)^{\perp}}{|a_i - a_j|^2}\cdot\nabla\chi(a_i),$$

which yields the conclusion. $\square$

Lemma 12. *Set $A = \cup B(a_i^0, 2r_a)\setminus B(a_i^0, r_a)$ and let $t_1, t_2 \in [0,T]$. Then for all $\varphi\in\mathcal{D}(A)$, we have*

$$\limsup_{k\to+\infty}\left|\int_{t_1}^{t_2}\int_A \left(\omega\left(u_{\varepsilon_k}(s)\right) - \omega\left(u^*(b_i(s), d_i)\right)\right)\varphi\right| \leq C\|\varphi\|_\infty \int_{t_1}^{t_2} h(s)\,ds.$$

Proof. We apply the pointwise equality (2.4) to $u \equiv u_{\varepsilon_k}(t)$ and $u^* \equiv u^*(b_i(t), d_i)$ for all t. Since $|u^*(b_i(t), d_i)| = 1$, this gives

$$\omega(u) - \omega(u^*) = \sum_{k,l=1}^{2}\Big(a_{k,l}\partial_l|u|\partial_k|u| + b_{k,l}\Big[\frac{j_k(u)}{|u|}\frac{j_l(u)}{|u|} - j_k(u^*)j_l(u^*)\Big]\Big),$$

where $a_{k,l}, b_{k,l}\in\mathbb{C}$. We rewrite the terms involving the components of j as

$$\frac{j_k(u)}{|u|}\frac{j_l(u)}{|u|} - j_k(u^*)j_l(u^*) = \Big(\frac{j_k(u)}{|u|} - j_k(u^*)\Big)\Big(\frac{j_l(u)}{|u|} - j_l(u^*)\Big)$$
$$+ j_k(u^*)\Big(\frac{j_l(u)}{|u|} - j_l(u^*)\Big) + j_l(u^*)\Big(\frac{j_k(u)}{|u|} - j_k(u^*)\Big).$$

We multiply the previous equality by φ, integrate on $[t_1,t_2]\times A$ and let k go to $+\infty$. Using the weak convergence in L^2 of $j(u_{\varepsilon_k})$ to $j(u^*(b_i(.), d_i))$ on $[0,T]\times A\subset\mathcal{G}$ combined with the fact that $ju^*(b_i(.), d_i)$ is bounded on this set, we deduce

$$\limsup_{k\to+\infty}\Big|\int_{t_1}^{t_2}\int_A\Big(\omega\left(u_{\varepsilon_k}(s)\right) - \omega\left(u^*(b_i(s), d_i)\right)\Big)\varphi\Big|$$
$$\leq c\|\varphi\|_\infty \limsup_{k\to+\infty}\int_{t_1}^{t_2}\int_A\Big(|\nabla|u_{\varepsilon_k}||^2 + \Big|\frac{ju_{\varepsilon_k}}{|u_{\varepsilon_k}|} - ju^*(b_i, d_i)\Big|^2\Big).$$

The conclusion finally follows from Lemmas 9 and 10. $\square$

We are now in position to complete the proof of Theorem 2. We consider arbitrary χ,φ belonging to $\mathcal{H}(a_i^0)$, we fix $0\leq s\leq t\leq T$ and we integrate the evolution formula (2.10) on $[s,t]$. We obtain

$$\int_s^t \frac{d}{d\tau}\int_{\mathbb{R}^2} Ju_{\varepsilon_k}(\tau)\chi + \delta\int_{\mathbb{R}^2}\mu_{\varepsilon_k}(\tau)\varphi = \int_s^t g_k^1(\tau) + \int_s^t g_k^2(\tau),$$

where

$$g_k^1(\tau) = -\delta\int_{\mathbb{R}^2}\frac{|\partial_t u_{\varepsilon_k}|^2}{|\log\varepsilon_k|^2} + R_{\varepsilon_k}(\tau,\chi,\varphi,u_{\varepsilon_k})$$

and

$$g_k^2(\tau) = 2\int_{\mathbb{R}^2} \mathrm{Im}\Big(\omega(u_{\varepsilon_k}(\tau))\frac{\partial^2\chi}{\partial\overline{z}^2}\Big),$$

which we decompose as

$$\begin{aligned} g_k^2 &= 2\int_{\mathbb{R}^2} \mathrm{Im}\left(\big[\omega(u_{\varepsilon_k}) - \omega(u^*(b_i,d_i))\big]\frac{\partial^2\chi}{\partial\overline{z}^2}\right) \\ &+ 2\int_{\mathbb{R}^2} \mathrm{Im}\left(\big[\omega(u^*(b_i,d_i)) - \omega(u^*(a_i,d_i))\big]\frac{\partial^2\chi}{\partial\overline{z}^2}\right) \\ &+ 2\int_{\mathbb{R}^2} \mathrm{Im}\left(\omega(u^*(a_i,d_i))\frac{\partial^2\chi}{\partial\overline{z}^2}\right) = A_k(\tau) + B_k(\tau) + C_k(\tau). \end{aligned}$$

We next substitute the formula given by Lemma 11 for C_k in the previous equalities. Setting

$$f_{k,\chi,\varphi}(\tau) = \int_{\mathbb{R}^2} Ju_{\varepsilon_k}(\tau)\chi + \delta\int_{\mathbb{R}^2}\mu_{\varepsilon_k}(\tau)\varphi - \pi\sum_{i=1}^{l}\Big(d_i\chi(a_i(\tau)) + \delta\varphi(a_i(\tau))\Big),$$

we obtain

$$f_{k,\chi,\varphi}(t) - f_{k,\chi,\varphi}(s) = \int_s^t g_1^k + \int_s^t A_k + \int_s^t B_k.$$

Lemma 6 with $T_\varepsilon = T$ first gives $|\int_s^t g_1^k(\tau)\,d\tau| \leq C|\log\varepsilon_k|^{-\frac{1}{2}}$ for all k. Moreover, it follows from Lemma 12 and the fact that supp $\frac{\partial^2\chi}{\partial\overline{z}^2} \subset A$ that

$$\limsup_{k\to+\infty}\Big|\int_s^t A_k(\tau)\,d\tau\Big| \leq C\int_s^t h(\tau)\,d\tau.$$

Finally, we infer from the regularity of $\omega(u^*)$ away from the vortices that

$$\int_s^t |B_k(\tau)|\,d\tau \leq C\int_s^t \sigma(\tau)\,d\tau \leq C\int_s^t h(\tau)\,d\tau.$$

Letting k go to $+\infty$, we finally deduce from the convergence statements in Theorem 5 that for $0 \leq s \leq t \leq T$,

$$|f_{\chi,\varphi}(t) - f_{\chi,\varphi}(s)| \leq C\int_s^t h(\tau)\,d\tau, \tag{6.3}$$

where $f_{\chi,\varphi}$ is defined by

$$f_{\chi,\varphi} = \pi\sum_{i=1}^{l}\Big[d_i\big(\chi(b_i) - \chi(a_i)\big) + \delta\big(\varphi(b_i) - \varphi(a_i)\big)\Big].$$

Here the constant C depends only on χ, φ and the initial conditions.

We now fix a time $t \in [0,T]$ at which all the vortices b_i have a time derivative. Since the a_i are C^1, it follows that $f_{\chi,\varphi}$ is differentiable at

t with time derivative given by

$$f'_{\chi,\varphi}(t) = \pi \sum_{i=1}^{l} \Big(d_i \nabla \chi(a_i^0) + \delta \nabla^\perp \chi(a_i^0) \Big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t) \big).$$

Dividing by $t - s$ in (6.3) and letting $s \to t$ gives then

$$\Big| \pi \sum_{i=1}^{l} \big(d_i \nabla \chi(a_i^0) + \delta \nabla^\perp \chi(a_i^0) \big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t) \big) \Big| \le C\, h(t).$$

So, considering in particular $\chi, \varphi \in \mathcal{H}(a_i^0)$ such that χ and φ vanish near each point a_i^0 except for one, we obtain for all $i = 1, \dots, l$

$$\Big| \pi \big(d_i \nabla \chi(a_i^0) + \delta \nabla^\perp \chi(a_i^0) \big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t) \big) \Big| \le C\, h(t).$$

Choosing then successively $\chi(x) = x_1$ and $\chi(x) = x_2$ near a_i^0 we end up with $|\dot{b}_i(t) - \dot{a}_i(t)| \le Ch(t)$, and it follows by summation

$$h'(t) \le Ch(t) \qquad \text{a.e.} \;\; t \in [0, T].$$

Since $h(0) = 0$, this implies that $h = 0$ on $[0, T]$, and hence $\sigma = 0$ on $[0, T]$. Applying Lemma 10, we infer that $\limsup_{k \to +\infty} \Sigma_{\varepsilon_k}(t) \le 0$. Besides, Lemma 3 yields for all $L \ge 2^{n_1}$

$$\begin{aligned} \liminf_{k \to +\infty} \Sigma_{\varepsilon_k}(t) &\ge \liminf_{k \to +\infty} \Big(\int_{B(L)} e_{\varepsilon_k}\left(u_{\varepsilon_k}(t)\right) - e_{\varepsilon_k}\left(u^*_{\varepsilon_k}(a_i(t), d_i)\right) \Big) - \frac{C}{L} \\ &\ge -\frac{C}{L}, \end{aligned}$$

where the second inequality is a consequence of the convergence of Jacobians on $B(L)$ stated in Lemma 8 (see [13, 17]). Letting L tend to $+\infty$, we obtain $\liminf_{k \to +\infty} \Sigma_{\varepsilon_k}(t) \ge 0$, so we deduce from (5.3) that $(u_{\varepsilon_k}(t))_{k \in \mathbb{N}}$ is well-prepared with respect to the configuration $(a_i(t), d_i)$. By uniqueness of the limit, this finally holds for the full family $(u_\varepsilon(t))_{0<\varepsilon<1}$ on $[0, T]$.

In conclusion, we observe that in our definition T only depends on K_1, r_a and $\max(R, R_a + r_a)$, so that we can extend our results to the whole of $[0, T^*)$ by repeating the previous arguments.

Appendix

We present here the proof of Theorem 1. We omit the dependence on ε and rewrite (2) in the following way

$$\begin{cases} \partial_t w = (a + ib)\big(\Delta w + f_{U_0}(w)\big), \\ w(0) = w_0 \in H^1(\mathbb{R}^2), \end{cases} \tag{CGL}$$

where

$$f_{U_0}(w) = \Delta U_0 + (U_0 + w)(1 - |U_0 + w|^2),$$

a is positive and $b \in \mathbb{R}$. We denote by $S = S(t,x)$ the semi-group operator associated to the corresponding homogeneous linear equation. Every solution to (CGL) satisfies the Duhamel formula

$$w(t,\cdot) = S(t,\cdot) * w_0 + \int_0^t \big(S(t-s,\cdot) * g_{U_0}(w(s),\cdot)\big)\, ds,$$

where $g_{U_0} = (a+ib)f_{U_0}$. The kernel S is explicitly given by

$$S(t,x) = \frac{1}{4\pi(a+ib)t} \exp\Big(\frac{-|x|^2}{4(a+ib)t}\Big).$$

Since a is positive, S decays at infinity like the standard heat Kernel. This will enable us to show that (CGL) enjoys the same smoothing properties as the parabolic Ginzburg-Landau equation. In particular, we have for all $1 \leq r \leq +\infty$ and for all $t > 0$

$$\|S(t,\cdot)\|_{L^r(\mathbb{R}^2)} \leq \frac{1}{t^{1-\frac{1}{r}}} \tag{a}$$

and concerning the space derivatives of $S(t)$,

$$\|D^k S(t,\cdot)\|_{L^r(\mathbb{R}^2)} \leq \frac{C(a,b)}{t^{\frac{|k|}{2}+1-\frac{1}{r}}}. \tag{b}$$

We will often use Young's inequality that gives for $f \in L^p(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2)$ $\|f*g\|_{L^r(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}\|g\|_{L^q(\mathbb{R}^2)}$, where $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. We first state local well-posedness for (CGL).

Proposition A.1. *Let $w_0 \in H^1(\mathbb{R}^2)$. Then there exists a positive time T^* depending on $\|w_0\|_{H^1}$ and a unique solution $w \in C^0([0,T^*), H^1(\mathbb{R}^2))$ to* (CGL)*.*

Proof. We intend to apply the fixed point theorem to the map $\psi : w \in H^1(\mathbb{R}^2) \mapsto \psi(w)$, where

$$\psi(w)(t) = S(t) * w_0 + \int_0^t S(t-s) * g_{U_0}(w(s))\, ds.$$

To this aim, we introduce $R = \|w_0\|_{H^1(\mathbb{R}^2)}$ and for $T > 0$

$$B(T,R) = \{w \in L^\infty([0,T], H^1(\mathbb{R}^2)) \text{ s.t. } \|w\|_{L^\infty(H^1)} \leq 3R\}.$$

We next show that we can choose $T = T(R)$ so that ψ maps $B(T(R),R)$ into itself and is a contraction on this ball.

For $T > 0$, we let $w \in B(T,R)$ and expand $f_{U_0}(w)$. Using that $H^1(\mathbb{R}^2)$ is continuously embedded in $L^p(\mathbb{R}^2)$ for all $2 \leq p < +\infty$ and the fact that U_0 belongs to $\mathcal{V}$, it can be shown that [5]

$$\|f_{U_0}\|_{L^\infty([0,T],L^2)} \leq C(U_0,R), \tag{c}$$

and for $w_1, w_2 \in B(T,R)$

$$\|f_{U_0}(w_1) - f_{U_0}(w_2)\|_{L^\infty([0,T],L^2)} \leq C(U_0,R)\|w_1 - w_2\|_{L^\infty([0,T],H^1)}. \tag{d}$$

[5] see Lemma 1 in [7].

We next apply Young's inequality to obtain

$$\begin{aligned}\|\psi(w)(t)\|_{H^1} &\leq \|\psi(w)(t)\|_{L^2} + \|\nabla\psi(w)(t)\|_{L^2} \\ &\leq 2\|S(t)\|_{L^1}\|w_0\|_{H^1} + \int_0^t \|S(t-s) + \nabla S(t-s)\|_{L^1}\|g_{U_0}(s)\|_{L^2}\, ds \\ &\leq 2\|w_0\|_{H^1} + C\int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right)\|g_{U_0}(w(s))\|_{L^2}\, ds,\end{aligned}$$

where the last inequality is a consequence of (a) and (b) with the choice $r = 1$. This yields according to (c) and (d)

$$\sup_{t\in[0,T]} \|\psi(w)(t)\|_{H^1} \leq 2\|w_0\|_{H^1} + C(U_0, R)(T + \sqrt{T})$$

and similarly,

$$\sup_{t\in[0,T]} \|\psi(w_1)(t) - \psi(w_2)(t)\|_{H^1} \leq C'(U_0, R)(T + \sqrt{T}) \sup_{t\in[0,T]} \|w_1(t) - w_2(t)\|_{H^1}.$$

The conclusion follows by choosing $T = T(R)$ sufficiently small so that $2\|w_0\|_{H^1} + C(U_0, R)(T + \sqrt{T}) \leq 3R$ and $C'(U_0, R)(T + \sqrt{T}) < 1$. □

We next show additional regularity for a solution to (CGL).

Lemma A.1. *Let $w \in C^0([0,T], H^1(\mathbb{R}^2))$ be a solution to* (CGL). *Then w belongs to $L^1_{\mathrm{loc}}([0,T], H^2(\mathbb{R}^2)) \cap C^0((0,T], H^2(\mathbb{R}^2))$ and therefore to $L^1_{\mathrm{loc}}([0,T], L^\infty(\mathbb{R}^2))$.*

Proof. We first differentiate $f_{U_0}(w)$ and use Lemma 2 in [7] which states by mean of various Sobolev embeddings, Hölder and Gagliardo-Nirenberg inequalities that

$$\partial_i f_{U_0}(w) = g_1(w) + g_2(w) \in L^\infty([0,T], L^2(\mathbb{R}^2)) + L^\infty([0,T], L^r(\mathbb{R}^2))$$

for all $1 < r < 2$. Moreover, we have

$$\sup_{s\in[0,T]} \|g_1(w)(s)\|_{L^2(\mathbb{R}^2)} + \|g_2(w)(s)\|_{L^r(\mathbb{R}^2)} \leq C(U_0, A(T), r),$$

where $A(T) = \sup_{s\in[0,T]} \|w(s)\|_{H^1(\mathbb{R}^2)}$. Next, differentiating twice Duhamel formula gives

$$\partial_{ij} w(t) = \partial_j S(t) * \partial_i w_0 + \int_0^t \partial_j S(t-s) * \partial_i f_{U_0}(s)\, ds,$$

so taking into account the decomposition $\partial_i f_{U_0} = g_1 + g_2$ we get

$$\begin{aligned}\|\partial_{ij} w(t)\|_{L^2} \leq \|\nabla S(t)\|_{L^1}\|\nabla w_0\|_{L^2} &+ \int_0^t \|\nabla S(t-s)\|_{L^1}\|g_1(s)\|_{L^2}\, ds \\ &+ \int_0^t \|\nabla S(t-s)\|_{L^\alpha}\|g_2(s)\|_{L^r}\, ds,\end{aligned}$$

where α is chosen so that $1+\frac{1}{2}=\frac{1}{\alpha}+\frac{1}{r}$. This finally yields in view of (b)

$$\|\partial_{ij}w(t)\|_{L^2} \leq \frac{C}{t^{\frac{1}{2}}}\|w_0\|_{H^1} + C(U_0, A(T), r)\int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}-1+\frac{1}{\alpha}}\right) ds.$$

Since $\frac{1}{2}+1-\frac{1}{\alpha}=\frac{1}{r}<1$, we conclude that the right-hand side is finite, so that $\partial_{ij}w(t) \in L^2(\mathbb{R}^2)$. □

Lemma **A**.1 enables to show that the renormalized energy is non-increasing and to establish a control of the growth of $\|w(t)\|_{H^1(\mathbb{R}^2)}$. For equation (CGL), this energy is given by

$$E_{U_0}(w)(t) = \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2} - \int_{\mathbb{R}^2} \Delta U_0 \cdot w + \int_{\mathbb{R}^2} \frac{(1-|U_0+w|^2)^2}{4}.$$

It is well-defined and continuous in time for $w \in C^0(H^1(\mathbb{R}^2))$.

Lemma A.2. *Let $w \in C^0([0,T), H^1(\mathbb{R}^2))$ be a solution to* (CGL). *Then for all $t \in (0,T)$ we have*

$$\frac{d}{dt}E_{U_0}(w)(t) \leq 0.$$

Moreover, there exists C depending only on $\|w_0\|_{H^1}$ and U_0 such that

$$\|w(t)\|_{H^1} \leq \|w_0\|_{H^1}\exp(Ct), \qquad \forall t \in [0,T). \tag{e}$$

Proof. We infer from equation (CGL) and Lemma **A**.1 that $\partial_t w$ belongs to $L^\infty_{\mathrm{loc}}((0,T], L^2(\mathbb{R}^2))$, so that we may compute

$$\begin{aligned}\frac{d}{dt}E_{U_0}(w(t)) &= \int_{\mathbb{R}^2} \nabla w\cdot\nabla\partial_t w - \Delta U_0\cdot\partial_t w - \partial_t w\cdot(U_0+w)(1-|U_0+w|^2)\\ &= -\int_{\mathbb{R}^2}\partial_t w\cdot(\Delta w + f_{U_0}(w))\\ &= -\int_{\mathbb{R}^2}\partial_t w\cdot(\frac{1}{a+ib}\partial_t w) = \frac{-a}{a^2+b^2}\int_{\mathbb{R}^2}|\partial_t w|^2 \leq 0.\end{aligned}$$

We now turn to (e). We compute for $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} w\cdot\partial_t w = \int_{\mathbb{R}^2} w\cdot[(a+ib)\Delta w] + \int_{\mathbb{R}^2} w\cdot[(a+ib)f_{U_0}(w)]\\ &= -a\int_{\mathbb{R}^2}|\nabla w|^2 + \int_{\mathbb{R}^2} w\cdot(a+ib)\Delta U_0\\ &\quad + \int_{\mathbb{R}^2} w\cdot[(a+ib)(U_0+w)(1-|U_0+w|^2)].\end{aligned}$$

We then split the last term in the previous equality as

$$\begin{aligned}\int_{\mathbb{R}^2} w\cdot[(a+ib)(U_0+w)(1-|U_0+w|^2)] &= \int_{\mathbb{R}^2} w\cdot[(a+ib)U_0(1-|U_0+w|^2)]\\ &\quad + a\int_{\mathbb{R}^2}|w|^2(1-|U_0+w|^2).\end{aligned}$$

The second term in the r.h.s. is clearly bounded by $a\|w(t)\|_{L^2(\mathbb{R}^2)}$. Using Cauchy-Schwarz inequality for the first one, we obtain

$$\int_{\mathbb{R}^2} w\cdot[(a+ib)(U_0+w)(1-|U_0+w|^2)] \leq C(U_0)\|w(t)\|_{L^2}V(t)^{\frac{1}{2}} + a\|w(t)\|_{L^2}^2,$$

where $V(t)=\int_{\mathbb{R}^2}(1-|U_0+w(t)|^2)^2$. We are led to

$$\frac{d}{dt}\|w(t)\|_{L^2(\mathbb{R}^2)}^2 \leq C(U_0)(\|w(t)\|_{L^2}^2+1+V(t)). \tag{f}$$

On the other hand, Cauchy-Schwarz inequality gives

$$E_{U_0}(w)(t) \geq \int_{\mathbb{R}^2}\frac{|\nabla w|^2}{2}\,dx - C(U_0)\|w(t)\|_{L^2} + \frac{V(t)}{4},$$

which yields, since E_{U_0} is non-increasing,

$$\frac{V(t)}{4} + \int_{\mathbb{R}^2}\frac{|\nabla w|^2}{2} \leq E_{U_0}(w_0) + C(U_0)\|w(t)\|_{L^2}. \tag{g}$$

We infer from (f) and (g)

$$\|w(t)\|_{L^2} \leq (1+\|w_0\|_{H^1})\exp(Ct)$$

and finally deduce (e) by using (g) once more. $\square$

Lemma **A**.2 provides global well-posedness for (CGL).

Proposition A.2. *Let $w_0\in H^1(\mathbb{R}^2)$. Then there exists a unique and global solution $w\in C^0(\mathbb{R}_+,H^1(\mathbb{R}^2))$ to* (CGL).

Proof. Let $w\in C^0([0,T^*),H^1(\mathbb{R}^2))$ be the unique maximal solution with initial condition w_0. If T^* is finite, we have according to (e)

$$\limsup_{t\to T^*}\|w(t)\|_{H^1(\mathbb{R}^2)} \leq C(U_0,T^*,w_0) < +\infty,$$

so that we can extend w to a solution $\overline{w}$ on $[0,T^*+\delta]$. This yields a contradiction. $\square$

We conclude this section with the following

Proposition A.3. *Let $w\in C^0(\mathbb{R}_+,H^1(\mathbb{R}^2))$ be the solution to* (CGL). *Then we have $w\in C^\infty(\mathbb{R}_+^*,C^\infty(\mathbb{R}^2))$.*

Proof. We proceed in several steps.

Step 1 Let $p\geq 2$ and $v\in H^p(\mathbb{R}^2)$. Then $D^k f_{U_0}(v)\in L^2(\mathbb{R}^2)+L^{\frac{4}{3}}(\mathbb{R}^2)$ for all $|k|\leq p$.

Proof of Step 1. We may assume in view of the proof of Lemma **A**.1 that $|k|\geq 2$. We decompose $f_{U_0}(v)$ as $f_{U_0}(v)=\Delta U_0+h_{U_0}(v)$, where

$$h_{U_0}(v)=(U_0+v)(1-|U_0+v|^2).$$

Since $U_0 \in \mathcal{V}$, it suffices to show that $D^k h_{U_0}(v) \in L^2(\mathbb{R}^2) + L^{\frac{4}{3}}(\mathbb{R}^2)$. Applying Leibniz's formula to $h_{U_0}(v)$, we obtain

$$\begin{aligned} D^k h_{U_0}(v) &= \sum_{m\le k} \binom{k}{m} D^{k-m}(U_0+v) D^m(1-|U_0+v|^2) \\ &= D^k(U_0+v) \\ &- \sum_{\substack{m\le k\\ n\le m}} \binom{k}{m}\binom{m}{n} D^{k-m}(U_0+v) D^n(U_0+v)\cdot D^{m-n}(U_0+v). \end{aligned}$$

Since $2 \le |k| \le p$, $v \in H^p(\mathbb{R}^2)$ and $U_0 \in \mathcal{V}$, we clearly have $D^k(U_0+v) \in L^2(\mathbb{R}^2)$.

For the second term in the right-hand side, we write each product inside the sum as

$$D^a(U_0+v) D^b(U_0+v)\cdot D^c(U_0+v)$$

with $|a|+|b|+|c| = |k| \ge 2$, and we examine all cases. We observe that $D^a(v+U_0)$ belongs to $H^1(\mathbb{R}^2)$ whenever $1 \le |a| \le p-1$ and hence to $L^4(\mathbb{R}^2)$, whereas $D^a(v+U_0)$ belongs to $L^2(\mathbb{R}^2)$ for $2 \le |a| \le p$. Since on the other hand $U_0 + v \in L^\infty$, we finally obtain

$$D^a(U_0+v) D^b(U_0+v)\cdot D^c(U_0+v) \in L^2(\mathbb{R}^2) + L^{\frac{4}{3}}(\mathbb{R}^2),$$

which yields the conclusion.

We now turn to the regularity in space for a solution to (CGL).

Step 2 Let $w \in C^0(\mathbb{R}_+, H^1(\mathbb{R}^2))$ be the solution to (CGL). Then for all $p \ge 1$ we have $w \in C^0(\mathbb{R}_+^*, H^p(\mathbb{R}^2))$.

Proof of Step 2. We proceed by induction on p. The case $p = 2$ has already been treated in Lemma **A**.1. Let us thus assume that $w \in C^0(\mathbb{R}_+^*, H^p(\mathbb{R}^2))$ for some $p \ge 2$. For $|k| \le p+1$, we differentiate $w(t)$ and we find

$$D^k w(t) = D^k(S(t) * w_0) + D^k \int_0^t S(t-s) * g_{U_0}(s)\, ds$$

which we rewrite as

$$\begin{aligned} D^k w(t) = D^k S(t) * w_0 &+ \int_0^{t/2} (D^k S(t-s)) * g_{U_0}(s)\, ds \\ &+ \int_{t/2}^t D^m S(t-s) * D^{k-m} g_{U_0}(s)\, ds, \end{aligned}$$

where m is a multi-index so that $|m| = 1$.

First, it follows from (b) that $t \mapsto D^k S(t) * w_0 \in C^0(\mathbb{R}_+^*, L^2(\mathbb{R}^2))$. Next, arguing that $g_{U_0} \in C^0(\mathbb{R}_+, L^2(\mathbb{R}^2))$ and using (b) with $r = 1$, we

find

$$\Big\| \int_0^{t/2} (D^k S(t-s)) * g_{U_0}(s)\, ds \Big\|_{L^2} \leq C \int_0^{t/2} \frac{ds}{(t-s)^{\frac{|k|}{2}}} \leq \frac{C}{t^{\frac{|k|}{2}-1}}.$$

On the other hand, since $|k-m| = |k|-1 \leq p$ and since by assumption $w(s) \in H^p(\mathbb{R}^2)$, Step 1 provides the decomposition

$$D^{k-m} g_{U_0}(s) = d^1(s) + d^2(s)$$

where d^1 belongs to $C^0(\mathbb{R}_+^*, L^2(\mathbb{R}^2))$ and d^2 to $C^0(\mathbb{R}_+^*, L^{\frac{4}{3}}(\mathbb{R}^2))$. It follows from (b) that

$$\begin{aligned}\Big\| \int_{t/2}^{t} D^m S(t-s) * D^{k-m} g_{U_0}(s)\, ds \Big\|_{L^2} &\leq \int_{t/2}^{t} \|\nabla S(t-s)\|_{L^1} \|d^1(s)\|_{L^2}\, ds \\ &\quad + \int_{t/2}^{t} \|\nabla S(t-s)\|_{L^r} \|d^2(s)\|_{L^{\frac{4}{3}}}\, ds \\ &\leq C(t) \int_{t/2}^{t} \Big((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}-1+\frac{1}{r}} \Big)\, ds,\end{aligned}$$

where r satisfies $1+\frac{1}{2} = \frac{1}{r} + \frac{3}{4}$. The last term is finite since $\frac{1}{2}+1-\frac{1}{r} = \frac{3}{4} < 1$, so we infer that $w \in C^0(\mathbb{R}_+^*, H^{p+1}(\mathbb{R}^2))$, as we wanted.

Step 3 Let $w \in C^0(\mathbb{R}_+, H^1(\mathbb{R}^2))$ be the solution to (CGL). Then we have $w \in C^k(\mathbb{R}_+^*, C^l(\mathbb{R}^2))$ for all $k, l \in \mathbb{N}$.

Proof of Step 3. For fixed $k, l \in \mathbb{N}$, we show by induction on $0 \leq j \leq k$ that $w \in C^j(\mathbb{R}_+^*, C^{l+2k-2j}(\mathbb{R}^2))$.

This holds for $j = 0$ according to Step 2 and to Sobolev embeddings. We assume next that $w \in C^j(\mathbb{R}_+^*, C^{l+2k-2j}(\mathbb{R}^2))$ for some $0 \leq j \leq k-1$, and it follows that

$$\Delta w,\ f_{U_0}(w) \in C^j(\mathbb{R}_+^*, C^{l+2k-2j-2}(\mathbb{R}^2)).$$

So, going finally back to equation (CGL), we obtain

$$w \in C^{j+1}(\mathbb{R}_+^*, C^{l+2k-2j-2}(\mathbb{R}^2)).$$

This concludes the proof of Proposition **A**.3.

□

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